K3 quotients of abelian surfaces in positive characteristic

Alvaro Gonzalez Hernandez

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy in Mathematics. It has been composed by myself and has not been submitted in any previous application for any degree.

The generative Artificial Intelligence tool ChatGPT has been used to spell check and edit parts of the text. The use of AI in this work adheres to the academic integrity framework on AI of the University of Warwick. In particular, generative AI has not been used to create original content, develop mathematical arguments, or perform any calculations.

Parts of this thesis have been made public:

1. *Explicit desingularisation of Kummer surfaces in characteristic two via specialisation.* To appear in the Journal of Symbolic Computation.

2. *Intersections of the automorphism and the Ekedahl-Oort strata in M_2.* Available on the arXiv.

Abstract

Kummer surfaces are K3 surfaces that arise as the quotients of abelian surfaces by the involution ι that sends each point to its inverse with respect to the group law. In an analogous way, one can construct K3 surfaces as quotients of abelian surfaces by the action of other finite groups. These are known as generalised Kummer surfaces.

In this thesis, we study classical and generalised Kummer surfaces, with a particular focus on the case of positive characteristic. We extend the current classification of generalised Kummer surfaces to characteristics two, three, and five, and construct many explicit examples, both when the abelian surface is the product of two elliptic curves and when it is the Jacobian of a genus two curve.

We also prove several related results, including the existence of Kummer surfaces with everywhere good reduction over number fields, and a description of the intersections between the automorphism strata and the Ekedahl–Oort strata inside the moduli space of genus two curves.

In chapter 1, we introduce the main properties of K3 surfaces and abelian varieties. We also review the theory of rational double points, and explain how to construct quotients of varieties from the perspective of Geometric Invariant Theory.

In chapter 2, we describe the current classification of generalised Kummer surfaces and extend it to characteristics two, three, and five.

In chapter 3, we explain how to construct explicit examples of generalised Kummer surfaces arising from abelian surfaces that are products of elliptic curves.

In chapter 4, we study the birational geometry of Kummer surfaces associated to Jacobian varieties of genus two curves, with a focus on fields of characteristic two. We also compute an example of a Kummer surface with everywhere good reduction over a quadratic number field.

In chapter 5, we compute the dimensions and number of irreducible components of the intersections between the automorphism strata and the pullback via the Torelli map of the Ekedahl–Oort strata inside the moduli space of genus two curves.

In chapter 6, we determine all possible groups of symplectic automorphisms of Jacobians of genus two curves that preserve the polarisation. We then compute the associated generalised Kummer surfaces given by the quotients by these groups.

1 Motivation and background

Since the very beginnings of algebraic geometry, one of its central aims has been the classification of geometric objects up to a suitable notion of equivalence. Over time, the field has developed a variety of topological and algebraic invariants for that purpose, such as for example, the dimension, the Betti numbers, or the Hodge numbers. These have been designed to help us distinguish between varieties and, when they are not isomorphic, to identify the structural properties they share. These and many other invariants provide the fundamental language through which one can organise the rich landscape of algebraic varieties.

In the modern approach, inspired by developments such as the minimal model program, classification often proceeds by relating varieties to one of three broad classes: Fano varieties, Calabi–Yau varieties, or varieties of general type. Each of these families plays a central role in the birational classification of higher-dimensional varieties, and studying their geometry allows us to answer many questions in arithmetic geometry and moduli theory.

Within this framework, K3 surfaces occupy a particularly distinguished position. As the two-dimensional Calabi–Yau varieties, they lie in a sweet spot: they are simple enough to be studied concretely, yet complex enough to exhibit a wide range of interesting geometric and arithmetic phenomena.

The aim of this thesis is to study a class of K3 surfaces known as generalised Kummer surfaces. These arise as quotients of abelian surfaces by the action of finite groups, and they serve as a rich source of explicit examples within the world of K3 surfaces.

While the classification of complex generalised Kummer surfaces has been known for decades, the focus of this thesis is on studying their behaviour over fields of positive characteristic. In this setting, when the characteristic of the base field divides the order of the acting group, we found new examples whose properties differ from those of their characteristic zero counterparts.

We will now introduce the concepts and techniques that will help us to understand these examples.

1.1 K3 surfaces

Definition 1.1.1. *A **K3 surface** is a smooth*[1] *surface X with trivial canonical bundle and $h^1(X, \mathcal{O}_X) = 0$.*

Some examples of K3 surfaces are

- Smooth quartics in $\mathbb{P}^3$.
- Smooth complete intersections of a quadric and a cubic in $\mathbb{P}^4$.
- Smooth complete intersections of three quadrics in $\mathbb{P}^5$.
- Smooth double coverings $X \to \mathbb{P}^2$ branched along a smooth sextic curve.

From the definition, one can easily see that all K3 surfaces have geometric genus $p_g(X) = 1$. By Serre duality and the definition of the Euler characteristic, one can also check that $\chi(X) = 2$ and from Noether's formula, we then deduce that $e(X) = 24$ and this gives us the Betti numbers

$$b_0(X) = 1, \qquad b_1(X) = 0, \qquad b_2(X) = 22, \qquad b_3(X) = 0, \qquad b_4(X) = 1.$$

Over $\mathbb{C}$, the Hodge diamond of a K3 surface is:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

One thing that stands out immediately from the Hodge diamond is that $h^{1,1}(X) = 20$, which suggests that K3 surfaces over $\mathbb{C}$ can support a wide range of divisor classes, as their Néron–Severi group can have rank as large as twenty. This contrasts with simpler examples of surfaces that we know, such as $\mathbb{P}^2$, whose Néron-Severi group has rank one.

1.1.1 The Néron-Severi lattice

Definition 1.1.2. *Let X be a smooth algebraic variety defined over an algebraically closed field. The **Picard group** $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X, with group operation given by tensor product.*

Let $\mathrm{Pic}^0(X)$ denote the connected component of $\mathrm{Pic}(X)$, i.e. the subgroup of line bundles that are algebraically equivalent to zero.

[1] When the context is clear, we will extend this definition to singular surfaces with rational double points (we will explain what these are later on) and sometimes say that a singular surface is a K3 surface if the desingularisation is a K3 surface.

Definition 1.1.3. *The **Néron–Severi group** of X, is the quotient*

$$\mathrm{NS}(X) = \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X).$$

For K3 surfaces, $\mathrm{NS}(X) = \mathrm{Pic}(X)$, as $\mathrm{Pic}^0(X) = h^1(X, \mathcal{O}_X) = 0$. As a consequence of the theorem of the base, we have the following result:

Theorem 1.1.4. *The Néron-Severi group of a smooth projective variety is a finitely generated abelian group* [GH94, Section 3.5].

As the Néron-Severi group is torsion-free, it has the structure of a lattice. In this thesis, we will mostly think of line bundles as Cartier divisors and addition in $\mathrm{Pic}(X)$ as formal sums of divisors. This is because in many instances, like in chapter 4, the line bundles we will work with will be linear combinations of effective divisors on X.

The rank of the Néron-Severi group of X is called the (geometric) **Picard number** and denoted by $\rho(X)$. In characteristic zero, from the fact that $h^{1,1}(X) = 20$, we deduce that the Picard rank must be less or equal than 20, whereas in positive characteristic, an upper bound is given by the second Betti number, so $\rho(X)$ can be 22 at most. All possible Picard numbers $1 \le \rho(X) \le 20$ can happen in any characteristic, whereas $\rho(X) = 21$ can never happen and $\rho(X) = 22$ only happens in positive characteristic.

There are many motivations as to why one would want to study the Néron-Severi group of a K3 surface. For instance, it helps us understand whether the surface admits elliptic fibrations or what its automorphism group is. Also, choosing a sublattice L of its Néron-Severi lattice gives it a way of rigidifying it, and we can construct moduli spaces of L-polarised K3 surfaces which parametrise K3 surfaces X with a primitive embedding $L \hookrightarrow \mathrm{NS}(X)$ [HT15].

A major challenge in the study of K3 surfaces is that, despite a good understanding of the general properties of their Néron–Severi lattices, working with individual examples remains very difficult. For instance, even for a smooth quartic surface in $\mathbb{P}^3$, the existing algorithms to compute the Picard number are highly inefficient, and determining the full Néron–Severi lattice is often entirely out of reach [vL07].

However, there is a set-up in which we have more control over the Néron-Severi lattice of a K3 surface V, which is when V is the desingularisation of a surface W with isolated singularities. In this case, the exceptional divisors introduced in the resolution of the singularities form a sublattice of $\mathrm{NS}(V)$ and thus contribute to the Picard number of V. Moreover, we can obtain detailed information about how the curves in the exceptional loci intersect, as we will now see.

1.2 Singular points on surfaces

1.2.1 Rational double points

We will recall the definition and main properties of rational double points following the exposition of the book *Algebraic Surfaces in Positive Characteristics*, by Miyanishi and Ito [MI20].

Let X be a normal projective surface with a unique singular point P and let the map $\pi : Y \to X$ be a resolution of singularities such that Y is smooth.

Let $E = \sum_{i=1}^{n} E_i$ be the exceptional locus of π. We say that a divisor $Z = \sum_{i=1}^{n} r_i E_i$ **has support in** E if $\mathrm{Supp}(Z) \subseteq E$. For such an effective divisor with support in E, we consider a subscheme $Z = (\mathrm{Supp}(Z), \mathcal{O}_Z)$ of Y, where $\mathcal{O}_Z$ is defined via the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-Z) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

The **Euler-Poincaré characteristic** $\chi(\mathcal{O}_Z)$ and the **arithmetic genus** $p_a(Z)$ are defined by

$$\chi\left(\mathcal{O}_Z\right) = h^0\left(\mathcal{O}_Z\right) - h^1\left(\mathcal{O}_Z\right), \qquad p_a(Z) = \frac{1}{2}\left(Z^2 + K_Y \cdot Z\right) + 1,$$

and by the Riemann-Roch theorem, we have that $\chi(\mathcal{O}_Z) = 1 - p_a(Z)$.

There exists a spectral sequence

$$E_2^{p,q} = H^p\left(X, R^q\pi_*\mathcal{O}_Y\right) \Longrightarrow H^{p+q}\left(Y, \mathcal{O}_Y\right)$$

Since $R^q\pi_*\mathcal{O}_Y = 0$ for $q > 1, \pi_*\mathcal{O}_Y = \mathcal{O}_X$, and $R^1\pi_*\mathcal{O}_Y$ is supported at the point P, we obtain the following long exact sequence

$$0 \to H^1\left(\mathcal{O}_X\right) \to H^1\left(\mathcal{O}_Y\right) \to H^0\left(R^1\pi_*\mathcal{O}_Y\right) \to H^2\left(\mathcal{O}_X\right) \to H^2\left(\mathcal{O}_Y\right) \to 0$$

It follows that

$$\chi\left(\mathcal{O}_X\right) - \chi\left(\mathcal{O}_Y\right) = h^0\left(R^1\pi_*\mathcal{O}_Y\right) \geq 0,$$

where $h^0\left(R^1\pi_*\mathcal{O}_Y\right) \geq p_a(Z)$ for any effective divisor Z supported in E.

This last inequality motivates the following definition:

Definition 1.2.1. *An isolated singular point P on a surface is a* ***rational singularity*** *if $R^1\pi_*\mathcal{O}_Y = 0$.*

Since $R^1\pi_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-sheaf supported at the point P, as we have seen, the vanishing of $h^0\left(R^1\pi_*\mathcal{O}_Y\right)$ is equivalent to the condition that $\chi\left(\mathcal{O}_Y\right) = \chi\left(\mathcal{O}_X\right)$.

This condition has been shown to be independent of the choice of resolution π. Hence, we may assume that π is **minimal**, in the sense that the exceptional locus $E = \pi^{-1}(P)$ contains no (-1)-curves, that is, no irreducible curves $E \cong \mathbb{P}^1$ with self-intersection $E^2 = -1$.

Consider the local ring $\mathcal{O}_{X,P}$, which has Krull dimension two, and let $\mathfrak{m}$ denote its maximal ideal. From the theory of Hilbert-Samuel polynomials, we know that the function $\dim_k(\mathcal{O}_{X,P}/\mathfrak{m}^{n+1})$ agrees with a quadratic polynomial for all sufficiently large n. In particular, this implies that when $n \gg 0$,

$$\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \mu n + \lambda$$

for some constants $\mu, \lambda \in \mathbb{Z}$. We refer to the constant $\mu(\mathcal{O}_{X,P})$ as the **multiplicity of *P***.

Definition 1.2.2. *An isolated singular point P on a surface X is a **rational double point** if P is a rational singularity and $\mu(\mathcal{O}_{X,P}) = 2$.*

The main property that makes us interested in studying rational double points on K3 surfaces is that they are the canonical singularities of surfaces:

Definition 1.2.3. *Let X be a normal variety with a singular point P such that its canonical divisor K_X is $\mathbb{Q}$-Cartier, and let $\pi : Y \to X$ be a resolution of the singularities of X. Then,*

$$K_Y = \pi^*(K_X) + \sum_{i=1}^{n} a_i E_i$$

*where the sum is over the irreducible exceptional divisors E_i and the a_i are rational numbers called **discrepancies**. Then, we say that P is a **canonical singularity** if $a_i \geq 0$ for all $1 \leq i \leq n$.*

Theorem 1.2.4 ([KM98, Theorem 4.20]). *A singular point on a surface is canonical if and only if it is a rational double point, in which case all the discrepancies are zero.*

This implies that if we have a surface X with only rational double points and we are able to prove that K_X is trivial, then, since theorem 1.2.4 implies that $K_Y = \pi^*(K_X)$, it follows that K_Y is also trivial. This is a key point that Katsura used to prove theorem 2.2.3.

Note that rational double points are known by several other names in the literature, such as **ADE singularities**, **du Val singularities**, or **Kleinian singularities**. These names reflect the different perspectives from which mathematicians have studied this class of singularities.

For instance, rational double points can also be understood in terms of:

- The configuration of the exceptional curves appearing in their minimal resolution.
- The analytic structure of the completed local ring $\widehat{\mathcal{O}}_{X,P}$.
- The fact that, in characteristic zero, they arise locally as quotients of $\mathbb{C}^2$ by a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

We will now explore how these different perspectives are connected.

Let $E = \sum_{i=1}^{n} E_i$ be the exceptional locus of π. Artin proved that if P is an isolated singular point then, the **intersection matrix**, which is the $n \times n$ matrix whose entries are $E_i \cdot X_j$, is negative-definite. Furthermore, there exists a smallest effective divisor $Z = \sum_{i=1}^{n} r_i E_i$ supported on E, such that $(Z \cdot E_i) \leq 0$ for all $1 \leq i \leq n$. This special divisor is known as the **fundamental cycle** of E.

He then proved the following results:

Theorem 1.2.5 ([Art66, Theorems 3 and 4]). *Let X be a normal projective surface with a singular point P, let $\pi : Y \to X$ be a resolution and E be the exceptional locus of π. Then,*

1. *P is a rational singularity if and only if the fundamental cycle Z of E satisfies that $p_a(Z) = 0$.*
2. $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = -(Z^2)n+1$. *In particular, the multiplicity of P is equal to $-(Z^2)$.*

The resolution of a singularity can be drawn as a graph where each irreducible curve C_i in the exceptional locus is drawn as a node, labelled by its self-intersection number C_i^2. Two nodes are then joined by an edge if and only if C_i and C_j intersect. To simplify notation, we omit the label on a node if $C_i^2 = -2$.

From theorem 1.2.5, we can deduce that if P is a rational double point, $(Z^2) = -2$. Using the negative-definiteness of the intersection matrix, du Val proved [DV34] that the desingularisation graph of a rational double point can be identified with one of the simply laced Dynkin diagrams:

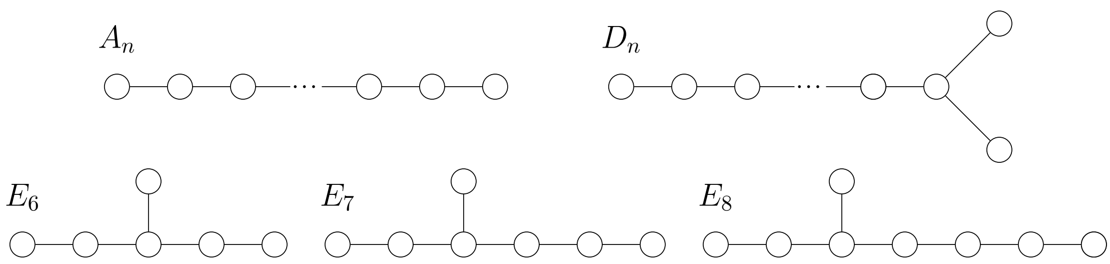

Figure 1.1: Simply laced Dynkin diagrams.

There is a deep and beautiful connection between lattices, surface singularities, and simple Lie groups, known as the **McKay correspondence**. It relates the ADE classification of simple Lie algebras with the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, which, as we will soon see, appear naturally in the study of rational double points.

Moreover, from theorem 1.2.5, we also deduce that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3$. Hence, if we let $\widehat{\mathcal{O}}_{X,P}$ denote the completion of $\mathcal{O}_{X,P}$, the Cohen structure theorem implies that

$$\widehat{\mathcal{O}}_{X,P} \cong k[[x,y,z]]/(f)$$

for some $f \in k[[x,y,z]]$.

This shows that rational double points are **hypersurface singularities**, meaning that X is locally isomorphic to a hypersurface defined by a single equation $f = 0$.

Rational double points in characteristic zero

Over an algebraically closed field of characteristic zero, every rational double point can, after a suitable change of coordinates, be written as a hypersurface singularity defined by one of the following equations:

$$\begin{aligned}
&A_n: && z^2 + x^2 + y^{n+1} = 0 && \text{for } n \geq 1,\\
&D_n: && z^2 + x^2y + y^{n-1} = 0 && \text{for } n \geq 4,\\
&E_6: && z^2 + x^3 + y^4 = 0, &&\\
&E_7: && z^2 + x^3 + xy^3 = 0, &&\\
&E_8: && z^2 + x^3 + y^5 = 0. &&
\end{aligned}$$

Another way to characterise rational double points over $\mathbb{C}$ is as quotients of the affine plane by finite subgroups $G \subset \mathrm{SL}_2(\mathbb{C})$ [Rei87]. The idea is that G acts linearly on $\mathbb{A}^2_{\mathbb{C}}$ via matrices of determinant one, and the singularity at the origin of the quotient space $\mathbb{A}^2/G$ is, in each case, a rational double point. In fact, the defining equations we have just seen can be recovered as relations among G-invariant polynomials on $\mathbb{A}^2$, according to the theory we will develop in section 1.5.1.

Some of the finite groups acting on surfaces in this context are non-standard, so it will be helpful to define them.

Definition 1.2.6. *The **binary dihedral group** (or dicyclic group) Q_{4n} of order $4n$ is the group defined by the presentation:*

$$Q_{4n} = \langle a, b \mid a^n b^2, b^4, abab^{-1} \rangle.$$

These groups can also be characterised as being non-split extensions of C_{2n} by a cyclic group of order two. The groups Q_8 and Q_{16} are known as the **quaternion groups** of orders 8 and 16.

As for the rest of them of order less than 24, we have that $Q_{12} \cong C_3 \rtimes C_4$, $Q_{20} \cong C_5 \rtimes C_4$, and $Q_{24} \cong C_3 \rtimes Q_8$. Another relevant family of groups that we need to know is the following:

Definition 1.2.7. *The **extended special linear group** $\mathrm{ESL}_2(\mathbb{F}_p)$ is the subgroup of $\mathrm{SL}_2(\mathbb{F}_{p^2})$ generated by $\mathrm{SL}_2(\mathbb{F}_p)$ and an element given by the diagonal matrix*

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

where $\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, and α^2 is a primitive element which generates $\mathbb{F}_p^\times$.

Remark 1.2.8. *Many of the groups acting on surfaces are **binary polyhedral groups**, that is, they are preimages under the map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of the group of rotations of a polyhedron. In addition to the binary dihedral groups, notable examples include: $\mathrm{SL}_2(\mathbb{F}_3)$, which is isomorphic to the **binary tetrahedral group**; $\mathrm{ESL}_2(\mathbb{F}_3)$, the **binary octahedral group**; and $\mathrm{SL}_2(\mathbb{F}_5)$, the **binary icosahedral group**.*

We then have the following classification:

Theorem 1.2.9. *Every rational double point is locally analytically isomorphic to the singularity at the point at the origin of a quotient $\mathbb{A}^2/G$, for some finite group $G \subset \mathrm{SL}_2(\mathbb{C})$. The correspondence is as follows:*

- *For the type A_n, $G = C_{n+1}$.*
- *For the type D_n, $G = Q_{4(n-2)}$.*
- *For the type E_6, $G = \mathrm{SL}_2(\mathbb{F}_3)$.*
- *For the type E_7, $G = \mathrm{ESL}_2(\mathbb{F}_3)$.*
- *For the type E_8, $G = \mathrm{SL}_2(\mathbb{F}_5)$.*

We have just seen that over $\mathbb{C}$, given a finite group $G \subset \mathrm{SL}_2(\mathbb{C})$, we can construct a rational double point as the singularity at the origin of the quotient $\mathbb{A}^2/G$. What is perhaps more surprising is that, given a rational double point on any surface over $\mathbb{C}$, we can always recover the group G from the topology of the surface.

Let P be a closed point of a normal surface X defined over an algebraically closed field, let $\mathcal{O}_{X,P}^h$ be the henselisation of $\mathcal{O}_{X,P}$ and $U = \mathrm{Spec}(\mathcal{O}_{X,P}^h)$. A **covering of U** is a finite surjective morphism $V \to U$ such that Y is irreducible and normal.

Definition 1.2.10. *The **local fundamental group** $\pi_1(U \setminus P)$ is the group that classifies finite coverings of U which are étale except above P.*

We then have the following result:

Theorem 1.2.11 ([Pri67, Theorem 3]). *Let P be a rational double point on a surface X, and let G be a finite group such that P is locally analytically isomorphic to the image of the origin in the quotient $\mathbb{A}^2/G$. Then $\pi_1(U \setminus P) = G$.*

Rational double points in positive characteristic

In characteristic p, it remains true that the resolution graph of a rational double point must correspond to one of the ADE Dynkin diagrams. However, unlike the situation in characteristic zero, in characteristics two, three, and five, there exist singularities with the same resolution graph that are not locally analytically isomorphic, as shown by Artin [Art75].

This raises a natural question: if two rational double points have the same configuration of exceptional curves in their resolutions, how can we tell whether they are locally analytically isomorphic?

One practical method is to compute an invariant known as the **Tjurina number**.

Definition 1.2.12. *Let X be a singular surface with a hypersurface singularity P, and let $\widehat{\mathcal{O}}_{X,P} \cong k[[x,y,z]]/(f)$. The **Tjurina number** at P is*

$$\tau(P) = \dim_k \frac{k[[x,y,z]]}{\langle f,\ \partial f/\partial x_i \rangle}.$$

In characteristic zero, the Tjurina numbers of the singularities of types A_n, D_n, and E_n are all equal to n. In positive characteristic, this is no longer true: the Tjurina number allows us to distinguish between rational double points that share the same resolution graph but are not locally analytically isomorphic.

Another major difference between the case over $\mathbb{C}$ and over a field k of positive characteristic is that not every rational double point arises as a quotient of the affine plane by the action of a finite group $G \subset \mathrm{SL}_2(\overline{k})$. For example, the singularity of type A_{p-1} in characteristic p cannot be realised as a quotient of $\mathbb{A}^2$ by the cyclic group C_p. This failure is reflected in the fact that the local fundamental group of a singularity of type A_{p-1} in characteristic p is trivial.

It is important to note, however, that many rational double points in positive characteristic are still quotient singularities, though not by the action of constant group schemes, but rather by infinitesimal group schemes, which we will introduce in subsection 1.3.3. Several examples can be found in the book by Miyanishi and Ito, including the construction of the A_{p-1} singularity in characteristic p as a quotient by the action of μ_p, and the construction of the singularity E_8^0 in characteristic five as a quotient by the action of α_5.

By studying the possible local fundamental groups of rational double points, Artin gave a complete classification of all rational double points in positive characteristic [Art75]:

If $p = 2$,

Type		Normal form	π_1	τ
A_n	$(n$ even$)$	$xy + z^{n+1}$	$(C_n)'$	n
A_n	$(n$ odd$)$	$xy + z^{n+1}$	C_n	$n+1$
D_{2n}^0	$(n \geq 2)$	$z^2 + x^2y + xy^n$	1	$4n$
D_{2n}^r	$(n \geq 2, 1 \leq r < \frac{n}{2})$	$z^2 + x^2y + xy^n + xy^{n-r}z$	1	$4n - 2r$
D_{2n}^r	$(n \geq 2, r = \frac{n}{2})$	$z^2 + x^2y + xy^n + xy^{n-r}z$	C_2	$4n - 2r$
D_{2n}^r	$(n \geq 2, \frac{n}{2} < r \leq n-1)$	$z^2 + x^2y + xy^n + xy^{n-r}z$	$D_{(2r-n)'}$	$4n - 2r$
D_{2n+1}^0	$(n \geq 2)$	$z^2 + x^2y + y^nz$	1	$4n$
D_{2n+1}^r	$(n \geq 2, 1 \leq r < \frac{n}{2})$	$z^2 + x^2y + y^nz + xy^{n-r}z$	1	$4n - 2r$
D_{2n+1}^r	$(n \geq 2, \frac{n}{2} < r \leq n-1)$	$z^2 + x^2y + y^nz + xy^{n-r}z$	$D_{4r-2n+1}$	$4n - 2r$
E_6^0		$z^2 + x^3 + y^2z$	C_3	8
E_6^1		$z^2 + x^3 + y^2z + xyz$	C_6	6
E_7^0		$z^2 + x^3 + xy^3$	1	14
E_7^1		$z^2 + x^3 + xy^3 + y^4z$	1	12
E_7^2		$z^2 + x^3 + xy^3 + x^2yz$	1	10
E_7^3		$z^2 + x^3 + xy^3 + xyz$	C_4	8
E_8^0		$z^2 + x^3 + y^5$	1	16
E_8^1		$z^2 + x^3 + y^5 + xyz$	1	14
E_8^2		$z^2 + x^3 + y^5 + y^3z$	C_2	12
E_8^3		$z^2 + x^3 + y^5 + y^2z$	1	10
E_8^4		$z^2 + x^3 + y^5 + xyz$	Q_{12}	8

Table 1.1: Rational double points in characteristic two.

If $p = 3$,

Type		Normal form	π_1	τ
A_n	$(3 \nmid (n+1))$	$z^2 + x^2 + y^{n+1}$	C_{n+1}	n
A_n	$(3 \mid (n+1))$	$z^2 + x^2 + y^{n+1}$	$(C_{n+1})'$	$n+1$
D_n	$(n \geq 4)$	$z^2 + x^2y + y^{n-1}$	$(Q_{4n-8})'$	n
E_6^0		$z^2 + x^3 + y^4$	1	6
E_6^1		$z^2 + x^3 + y^4 + x^2y^2$	C_3	7
E_7^0		$z^2 + x^3 + xy^3$	C_2	9
E_7^1		$z^2 + x^3 + xy^3 + x^2y^2$	C_6	7
E_8^0		$z^2 + x^3 + y^5$	1	12
E_8^1		$z^2 + x^3 + y^5 + x^2y^3$	1	10
E_8^2		$z^2 + x^3 + y^5 + x^2y^2$	$\mathrm{SL}_2(\mathbb{F}_3)$	8

Table 1.2: Rational double points in characteristic three.

If $p = 5$,

Type	Normal form	π_1	τ
A_n $(5 \nmid (n+1))$	$z^2 + x^2 + y^{n+1}$	C_{n+1}	n
A_n $(5 \mid (n+1))$	$z^2 + x^2 + y^{n+1}$	$(C_{n+1})'$	$n+1$
D_n $(n \geq 4)$	$z^2 + x^2y + y^{n-1}$	$(Q_{4n-8})'$	n
E_6	$z^2 + x^3 + y^4$	$\mathrm{SL}_2(\mathbb{F}_3)$	6
E_7	$z^2 + x^3 + xy^3$	$\mathrm{ESL}_2(\mathbb{F}_3)$	7
E_8^0	$z^2 + x^3 + y^5$	1	10
E_8^1	$z^2 + x^3 + y^5 + xy^4$	C_5	8

Table 1.3: Rational double points in characteristic five.

If $p \geq 7$,

Type	Normal form	π_1	τ
A_n $(p \nmid (n+1))$	$z^2 + x^2 + y^{n+1}$	C_{n+1}	n
A_n $(p \mid (n+1))$	$z^2 + x^2 + y^{n+1}$	$(C_{n+1})'$	$n+1$
D_n $(n \geq 4)$	$z^2 + x^2y + y^{n-1}$	$(Q_{4n-8})'$	n
E_6	$z^2 + x^3 + y^4$	$\mathrm{SL}_2(\mathbb{F}_3)$	6
E_7	$z^2 + x^3 + xy^3$	$\mathrm{ESL}_2(\mathbb{F}_3)$	7
E_8	$z^2 + x^3 + y^5$	$\mathrm{SL}_2(\mathbb{F}_5)$	8

Table 1.4: Rational double points in characteristic $p \geq 7$.

In these tables, G' denotes the maximal prime-to-p quotient of G and $D_{(2r-n)'}$ is the dihedral group of order $2m$ where m is the greatest divisor of $(2r - n)$ which is not divisible by two.

We will use the information of these tables in chapter 2 to study when the quotient of an abelian surface by a finite group has only rational double points.

1.2.2 Elliptic singularities

As we have seen, rational double points behave exceptionally well under resolution: they do not alter the canonical bundle of the surface. This property is quite rare among surface singularities. For instance, consider the case of elliptic singularities:

Definition 1.2.13. *Let X be a normal projective surface with a singular point P, let $\pi : Y \to X$ be a resolution, E the exceptional locus of π and Z the fundamental cycle of E. We say that P is an **elliptic singularity** if $p_a(Z) = 1$.*

We can check that whenever we have an elliptic singularity, the discrepancies of the resolution $\pi : Y \to X$ are not all zero and, therefore, $K_Y \neq \pi^*(K_X)$. For example, while the resolution of a quartic surface with only rational double points is always a K3 surface, this is no longer true if the surface has elliptic singularities: the minimal resolution may be rational, as shown by Katsura [Kat78, Proposition 9].

While rational double points admit a neat classification via the ADE families, the classification of elliptic singularities is significantly more intricate. There is a general classification due to Wagreich [Wag70], as well as a more refined classification for a special subclass known as **minimally elliptic singularities**, due to Laufer [Lau77]. We will make use of both classifications in this thesis. However, we note that in practice, it is often difficult to determine whether a singularity is elliptic solely from its desingularisation.

In chapters 3 and 4, we will examine several examples of surfaces with elliptic singularities.

1.3 Abelian varieties

In this thesis, we will construct K3 surfaces as quotients of abelian surfaces. It is therefore important to introduce some foundational concepts from the theory of abelian varieties. Many of the definitions in this section follow the notes *Geometry and arithmetic of moduli spaces of abelian varieties in positive characteristic* by Karemaker [Kar24].

Definition 1.3.1. *An **abelian variety** is a connected and complete group variety.*

There are two standard constructions of abelian varieties that we will use throughout this thesis:

1. The product of n elliptic curves is an abelian variety of dimension n.
2. Given a smooth projective curve $\mathcal{C}$ of genus g, we can construct its **Jacobian variety** $\mathrm{Jac}(\mathcal{C})$, which is the connected component of the identity of the Picard scheme $\mathrm{Pic}^0_{\mathcal{C}/k}$. It is an abelian variety of dimension g, and its k-points correspond to algebraically trivial line bundles on $\mathcal{C}$.

These two types of examples are often related by **isogenies**, which are homomorphisms $f : A \to A'$ of abelian varieties that are finite, flat and surjective. We say A and A' are **isogenous** if there exists such an isogeny and we denote this by $A \sim A'$.

While abelian varieties are projective, they are not canonically embedded in projective space. In order to understand projective embeddings of abelian varieties, it is important to study its ample line bundles, which leads naturally to the notion of the dual abelian variety.

Definition 1.3.2. *Let A be an abelian variety defined over a field k. The **dual abelian variety** $A^\vee = \mathrm{Pic}^0_{A/k}$ is the connected component of the identity of the Picard scheme.*

Similarly to the Jacobian, $A^\vee$ is an abelian variety whose k-points correspond to algebraically trivial line bundles on A, i.e. $A^\vee(k) = \mathrm{Pic}^0(A)$.

Given an ample line bundle L on A, one can associate an isogeny

$$\varphi_L\colon \quad \begin{aligned} A &\longrightarrow A^\vee \\ x &\longmapsto [t_x^* L \otimes L^{-1}], \end{aligned}$$

where the square brackets denote the isomorphism class of the resulting line bundle.

Definition 1.3.3. *A **polarisation** of an abelian variety A defined over k is an isogeny $\lambda : A \to A^\vee$ such that there exists a finite field extension k'/k and an ample line bundle L on $A \times_k k'$ with the property that $\lambda_{k'} = \varphi_L$.*

Any abelian variety A admits a polarisation of some degree. A polarisation λ is called **principal** if it is an isomorphism; in this case, we say that A is **principally polarised**. One of the reasons why principal polarisations are useful is to be able to construct well-behaved moduli spaces of abelian varieties.

Given the Jacobian of a curve of genus g, there is also always a way of endowing it with a principal polarisation by considering the polarisation associated to its theta divisor. We will elaborate more on how this can be achieved when $g = 2$ in section 4.2.

Moreover, if we have a product of principally polarised abelian varieties, we can obtain a principal polarisation on the product by taking the tensor product of the pullbacks of the ample line bundles defining the polarisations from each factor via the projection maps. This product polarisation is one method of constructing a principal polarisation, but it is important to note that a given abelian variety may admit several non-isomorphic principal polarisations. However, this ambiguity is often manageable: in the case of abelian surfaces, for instance, there is a concrete classification of all possible principally polarised abelian surfaces.

Theorem 1.3.4 ([Wei57, Satz 2]). *Every principally polarised abelian surface is isomorphic to either the Jacobian variety of a genus two curve or the product of two elliptic curves with the product polarisation.*

1.3.1 The endomorphism algebra

Definition 1.3.5. *An **endomorphism** is a homomorphism from an abelian variety to itself. All endomorphisms of a fixed abelian variety form a ring under addition and composition and we denote the endomorphism ring of A by $\mathrm{End}(A)$.*

For any A over k and any integer n, we have the endomorphism $[n] : A \to A$ given on points by $x \mapsto nx$. Thus, for any A we see that $\mathbb{Z} \hookrightarrow \mathrm{End}(A)$. The degree of $[n]$ is $n^{2\dim(A)}$ for any n. We will denote by $A[n]$ the kernel of the previous map and the points of $A[n](k)$, we will refer to as the ***n*-torsion points**.

If an isogeny $f : A \to A'$ has degree n, then there exists an isogeny $g : A' \to A$ which satisfies $f \circ g = g \circ f = [n]$.

The **endomorphism algebra** of A is $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If A is **simple**, that is, it contains no non-trivial abelian subvarieties, then $\mathrm{End}^0(A)$ is a division algebra. Indeed, any non-zero endomorphism $f \in \mathrm{End}(A)$ of degree n is invertible in $\mathrm{End}^0(A)$, with inverse given by $\frac{1}{n}g$ as above.

Furthermore, recall that any abelian variety A admits a polarisation λ of some degree. This implies that the endomorphism algebra $\mathrm{End}^0(A)$ always contains a positive involution $\alpha \mapsto \lambda^{-1} \circ \alpha^\vee \circ \lambda$, called the **Rosati involution.**

Combining these two facts, we deduce that the endomorphism algebra of a simple abelian variety is a division algebra equipped with a positive involution. Such objects were completely classified by Albert [Alb30], and from his classification, we obtain the following result:

Theorem 1.3.6. *The endomorphism algebra of a simple abelian variety A over an algebraically closed field is isomorphic to one of the following:*

- ***Type I.*** *A totally real field F of degree dividing* $\dim(A)$.
- ***Type II.*** *A totally indefinite quaternion division algebra (i.e. split at each real place) over a totally real field F.*
- ***Type III.*** *A totally definite quaternion division algebra (i.e. non-split at each real place) over a totally real field F.*
- ***Type IV.*** *A central division algebra whose centre is a CM-field, i.e. a totally imaginary quadratic extension of a totally real field F.*

For elliptic curves E we have that $\mathrm{End}(E)$ is either $\mathbb{Z}$, an order in a quadratic imaginary field $\mathbb{Q}(\sqrt{-d})$ or an order in a quaternion algebra $\mathbb{H}_{p,\infty}$. Therefore, $\mathrm{End}^0(E)$ is isomorphic to either $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-d})$ or $\mathbb{H}_{p,\infty}$. In characteristic zero, the endomorphism algebra is necessarily commutative, so only the first two cases can occur. The quaternionic case arises only in characteristic p when E is supersingular. In that case, $\mathrm{End}^0(E) \cong \mathbb{H}_{p,\infty}$, the unique definite quaternion algebra over $\mathbb{Q}$ which only ramifies at p and at infinity.

If an abelian variety A is not simple, then up to isogeny it decomposes as

$$A \sim A_1^{s_1} \times \cdots \times A_r^{s_r}$$

where the A_i are pairwise non-isogenous simple abelian varieties which are uniquely determined up to isogeny. In this case, the endomorphism algebra decomposes accordingly as

$$\mathrm{End}^0(A) = M_{s_1}(\mathrm{End}^0(A_1)) \oplus \cdots \oplus M_{s_r}(\mathrm{End}^0(A_r))$$

where $M_n(B)$ denotes the algebra of $n \times n$ matrices over B.

Combining these results, we deduce the following:

Corollary 1.3.7. *The endomorphism algebra of an abelian surface A over an algebraically closed field of characteristic $p \geq 0$ is isomorphic to one of the following:*

- *If A is simple, $\mathrm{End}^0(A)$ can be isomorphic to*
  - *The rationals.*
  - *A real quadratic field.*
  - *A totally imaginary quadratic extension of a real quadratic field.*
  - *A quaternion division algebra over $\mathbb{Q}$.*
  - *A quaternion division algebra over an imaginary quadratic field.*
- *If A is not simple, $\mathrm{End}^0(A)$ can be isomorphic to*
  - $\mathbb{Q} \oplus \mathbb{Q}$.
  - $\mathbb{Q} \oplus \mathbb{Q}(\sqrt{-d'})$, *for some $d' > 0$.*
  - $\mathbb{Q}(\sqrt{-d}) \oplus \mathbb{Q}(\sqrt{-d'})$, *for some $d, d' > 0$.*
  - $\mathbb{Q}(\sqrt{-d'}) \oplus \mathbb{H}_{p,\infty}$, *for some $d' > 0$.*
  - $M_2(\mathbb{Q})$.
  - $M_2(\mathbb{Q}(\sqrt{-d'}))$.
  - $M_2(\mathbb{H}_{p,\infty})$.

Knowing about the endomorphism algebra of an abelian surface will be very helpful to be able to determine what the possible finite groups acting on it are, as we will see in the next chapter.

1.3.2 The Tate module

Let A be an abelian variety defined over a perfect field k of characteristic $p \geq 0$.

Definition 1.3.8. *The **Tate module** of A at a prime ℓ is defined as the inverse limit*

$$T_\ell(A) = \varprojlim A[\ell^r](\bar{k}),$$

where the transition maps are given by multiplication by ℓ:

$$A[\ell^{r+1}](\bar{k}) \xrightarrow{\times \ell} A[\ell^r](\bar{k}).$$

*The **rational Tate module** is defined as $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.*

When $\ell \neq p$, the module $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of rank $2\dim(A)$.

The restriction $\ell \neq p$ is essential. When A is defined over a field of characteristic p, the behaviour of its p^r-torsion subgroup is more subtle. Over an algebraically closed field, one usually has that

$$A[\ell^r](\bar{k}) \cong (\mathbb{Z}/\ell^r\mathbb{Z})^{2\dim(A)}.$$

However, this is no longer true when $\ell = p$. Instead, one has

$$A[p^r](\bar{k}) \cong (\mathbb{Z}/p^r\mathbb{Z})^{f(A)},$$

where $f(A)$ is an integer between 0 and $\dim(A)$ known as the ***p*-rank** of A.

Definition 1.3.9. *An abelian variety A defined over a field of characteristic p is called* ***ordinary*** *if $f(A) = \dim(A)$.*

In contrast to ordinary abelian varieties, we have supersingular abelian varieties.

Definition 1.3.10. *An abelian variety A defined over a field k of characteristic p is* ***supersingular*** *if it is isogenous over $\bar{k}$ to a product of elliptic curves*

$$A \sim E_1 \times \cdots \times E_n$$

such that $f(E_i) = 0$ for all $1 \leq i \leq n = \dim(A)$.

If A is not only isogenous to a product of supersingular elliptic curves but isomorphic to it, we say that A is **superspecial**.

It is easy to see that if A is supersingular, then $f(A) = 0$. However, in general, $f(A) = 0$ does not imply that A is supersingular. This implication does hold when $\dim(A) \leq 2$, but often fails in higher dimensions. For instance, in dimension ≥ 3, there exist abelian varieties with p-rank zero that are not isogenous to a product of supersingular elliptic curves. This can be seen through the theory of Ekedahl-Oort types, which we will discuss very briefly in section 5.4.

The p-rank of an abelian variety carries significant geometric and arithmetic information, as we will see throughout this thesis. One fruitful way to understand this phenomenon is through the theory of finite flat group schemes.

1.3.3 Group schemes and the Dieudonné module

Definition 1.3.11. *A* ***group scheme*** *G over S is a group object in the category Sch_S of schemes over S. That is, G is equipped with morphisms*

$$m : G \times_S G \to G \quad \textit{(multiplication)}, \qquad e : S \to G \quad \textit{(identity)}, \qquad i : G \to G \quad \textit{(inverse)},$$

satisfying the usual group axioms via commutative diagrams. We say that G is an ***affine group scheme*** *if it is representable by an affine scheme, that is, $G = \mathrm{Spec}(A)$ for some k-algebra A.*

We now describe some examples of group schemes that will appear throughout this thesis.

The multiplicative group scheme

The **multiplicative group scheme** $\mathbb{G}_m$ is defined to be

$$\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}]).$$

If $T = \mathrm{Spec}(R)$, then, $\mathbb{G}_m(T) = R^\times$ and the multiplication map is given by

$$\begin{aligned} m(T)\colon \quad \mathbb{G}_m(T) \times \mathbb{G}_m(T) &\longrightarrow \mathbb{G}_m(T) \\ (x, y) &\longmapsto xy \end{aligned}$$

The constant group scheme

Let G be a group. The **constant group scheme** $\underline{G}$ is defined to be

$$\underline{G} = \bigsqcup_{g \in G} \mathrm{Spec}(k).$$

Then, for any scheme T, $\underline{G}(T) = \bigsqcup_{g \in G} T$. This implies that $\underline{G}(T) \times \underline{G}(T) = \bigsqcup_{(g_1, g_2) \in G \times G} T$ and the multiplication map $m : \underline{G}(T) \times \underline{G}(T) \to \underline{G}(T)$ is defined by mapping the component labelled by (g_1, g_2) to the component labelled by $g_1 g_2 \in G$.

The group scheme μ_n

The **group scheme** μ_n is defined to be

$$\mu_n = \mathrm{Spec}(k[t]/(t^n - 1)).$$

If $T = \mathrm{Spec}(R)$, then, $\mu_n(T) = \{x \in R : x^n = 1\}$ and the multiplication map is given by

$$\begin{aligned} m(T)\colon \quad \mu_n(T) \times \mu_n(T) &\longrightarrow \mu_n(T) \\ (x, y) &\longmapsto xy \end{aligned}$$

Whenever all the primitive n-th roots of unity are defined over k and $p = \mathrm{char}(k)$ satisfies that $p \nmid n$, one can check that $\mu_n \cong \underline{\mathbb{Z}/n\mathbb{Z}}$. However, when $p \nmid n$, for instance, these are not étale and hence not isomorphic to the constant group scheme. Furthermore, in characteristic p there are more group schemes of order p:

The group scheme α_p

Let k be a field of characteristic p. The **group scheme** α_p is defined to be

$$\alpha_p = \mathrm{Spec}(k[t]/(t^p)).$$

This is a subgroup of the additive group scheme $\mathbb{G}_a$ and, therefore, when $T = \mathrm{Spec}(R)$, then, $\alpha_p(T) = \{x \in R : x^p = 0\}$ and the multiplication map is given by

$$\begin{aligned} m(T)\colon \quad \alpha_p(T) \times \alpha_p(T) &\longrightarrow \alpha_p(T) \\ (x, y) &\longmapsto x + y \end{aligned}$$

When $\mathrm{char}(k) = p$, we have seen that the classification of finite group schemes becomes significantly richer. This is due to the presence of the Frobenius morphism, which introduces additional structure not present in characteristic zero. A particularly useful tool for studying such group schemes is the theory of Dieudonné rings.

Let $W(k)$ denote the ring of Witt vectors over a perfect field k, and we will denote by $\sigma : W(k) \to W(k)$ the canonical lift of the Frobenius automorphism of k.

Definition 1.3.12. *The* ***Dieudonné ring*** D_k *is the non-commutative ring generated over* $W(k)$ *by two formal operators* F *and* V*, subject to the relations*

$$FV = VF = p, \quad Fa = \sigma(a)F, \quad aV = V\sigma(a) \quad \text{for all } a \in W(k).$$

This ring allows for the classification of finite commutative group schemes of p-power order:

Theorem 1.3.13 ([Fon77, Théorème 1]). *Suppose* k *is perfect. There is a contravariant equivalence of categories* $\mathrm{M}(-)$ *between the category of finite commutative group schemes over* k *of* p*-power order and the category of left* D_k*-modules of finite length.*

Let A be an abelian variety over a field of characteristic p. Its associated ***p*-divisible group** $A[p^\infty]$ is defined as the direct limit

$$A[p^\infty] \coloneqq \varinjlim A[p^n],$$

with respect to the natural inclusions $A[p^n] \hookrightarrow A[p^{n+1}]$.

Definition 1.3.14. *The* ***Dieudonné module of*** A *is defined to be*

$$\mathrm{M}(A) \coloneqq \mathrm{M}(A[p^\infty]).$$

The module $\mathrm{M}(A)$ is free of rank $2\dim(A)$ over $W(k)$.

Note that in the literature, the notation $T_p(A)$ is sometimes used to refer to the Dieudonné module. However, in this thesis we reserve $T_p(A)$ for the p-adic Tate module, as defined in definition 1.3.8. From the comparison above, we see that, unlike $\mathrm{M}(A)$, which is a free $W(k)$-module of rank $2\dim(A)$, $T_p(A)$ is a free $\mathbb{Z}_p$-module of rank equal to the p-rank of A. When $k = \mathbb{F}_p$, one can relate $T_p(A)$ to the étale part of $\mathrm{M}(A)$, though this requires careful technical treatment [Fon77].

1.4 Kummer surfaces: bridging the gap between abelian and K3 surfaces

The connection between abelian and K3 surfaces was first explored in the nineteenth century by mathematicians such as Göpel, Kummer and Borchardt, while studying the remarkable geometry of quartic surfaces in $\mathbb{P}^3$ with sixteen singularities.

They discovered that these quartic surfaces, which are examples of K3 surfaces, can always be constructed as the quotient of the Jacobian of a genus two curve by the natural involution on $\mathrm{Jac}(\mathcal{C})$ that sends each point to its inverse under the group law. This motivates the following definition:

Definition 1.4.1. *Let A be an abelian surface defined over a field of characteristic not equal to two. The **Kummer surface** $\mathrm{Kum}(A)$ is the quotient of A by the involution*

$$\iota\colon \quad \begin{aligned} A &\longrightarrow A \\ P &\longmapsto -P \end{aligned}$$

The details of this construction will be discussed in chapter 4, where we will explain in detail many aspects of the construction of Kummer surfaces, including why we have assumed that we are not working over a field of characteristic two. For now, let us highlight some of their key geometric properties, which illustrate why Kummer surfaces are among the most important examples of K3 surfaces.

The involution ι on A has exactly sixteen fixed points, corresponding to the 2-torsion points of A. On the quotient surface, these become singularities locally isomorphic to the quotient of $\mathbb{A}^2$ by the action of C_2, and are therefore A_1 singularities. As a result, the minimal resolution Y of $\mathrm{Kum}(A)$ contains sixteen disjoint (-2)-curves. Since the class of a hyperplane section is linearly independent of these, we deduce that the Picard number satisfies $\rho(Y) \geq 17$.

Moreover, the quotient map $A \to \mathrm{Kum}(A)$ allows us to relate the Néron–Severi groups of A and Y. In particular, once we understand $\mathrm{NS}(A)$, which is typically easier thanks to the rich theory of abelian varieties, we can often deduce information about $\mathrm{NS}(Y)$, such as its rank, discriminant, elliptic fibrations, and automorphism group.

Another area in which the connection to abelian varieties is particularly useful is in studying the reduction of $\mathrm{Kum}(A)$ at primes. Through the Néron-Ogg-Shafarevich criterion, one can analyse the reduction of an abelian surface at a prime from the ramification of its Tate module, and use this information to study the reduction of the Kummer. We will apply this theory in section 4.7, where we will find explicit examples of Kummer surfaces defined over a number field that have good reduction at all primes.

Finally, another reason why Kummer surfaces are important is that they allow us to perform explicit computations with Jacobians of genus two curves. In section 4.2, we will see that embedding $\mathrm{Jac}(\mathcal{C})$ into projective space requires 72 quadrics in $\mathbb{P}^{15}$.

Since $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ admits a model as a quartic surface in $\mathbb{P}^3$, and there is a 2-to-1 cover $\mathrm{Jac}(\mathcal{C}) \to \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$, Kummer surfaces serve as a practical intermediary for computations involving Jacobians. This makes them particularly useful in number theory, where they are employed to compute rational points on genus two curves and their Jacobians [CF96], and in cryptography, where they play an important role in genus two isogeny-based protocols [FT19].

In order to understand Kummer surfaces well, it is essential to be comfortable with working with quotient varieties. In the next section, we will give a brief introduction to the study of quotients of varieties by algebraic group actions, and how to construct defining equations for such quotients explicitly.

1.5 Constructing GIT quotients

The area of Geometric Invariant Theory is quite vast. In this thesis, we will briefly summarise the foundational results needed to construct quotients of varieties by group actions. The material presented here is based on the notes on *Moduli problems and Geometric Invariant Theory* by Hoskins [Hos15].

1.5.1 Affine GIT

Let G be a group acting on a variety X. The main goal of Geometric Invariant Theory is to study the geometric structure of the **orbit space**:

$$X/G = \{G \cdot x : x \in X\}.$$

Let G be an **algebraic group**, that is, a group scheme over $\mathrm{Spec}(k)$ for some field k. We say that G is an **affine algebraic group** if the underlying scheme of G is affine.

Suppose G is an affine algebraic group acting on two schemes X and Y, via actions $\sigma_X : G \times X \to X$ and $\sigma_Y : G \times Y \to Y$, respectively. A morphism $f : X \to Y$ is said to be ***G*-equivariant** if the following diagram commutes:

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\sigma_X"] & G \times Y \arrow[d, "\sigma_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If Y is equipped with the **trivial action** $\pi_Y : G \times Y \to Y$, then a G-equivariant morphism $f : X \to Y$ is called a ***G*-invariant morphism**.

There are several notions of quotients in algebraic geometry. We begin with the most universal one:

Definition 1.5.1. *A **categorical quotient** for the action of G on X is a G-invariant morphism $\varphi : X \to Y$ of schemes which is universal; that is, every other G-invariant morphism $f : X \to Z$ factors uniquely through φ so that there exists a unique morphism $h : Y \to Z$ such that $f = h \circ \varphi$.*

Let G be an affine algebraic group acting on a scheme X over k. The group G acts on the k-algebra $\mathcal{O}_X(U)$ of regular functions on a G-invariant subset $U \subset X$ by

$$g \cdot f(x) = f\left(g^{-1} \cdot x\right).$$

We denote the subalgebra of ***G*-invariant functions** by

$$\mathcal{O}_X(U)^G \coloneqq \{f \in \mathcal{O}_X(U) : g \cdot f = f \text{ for all } g \in G\}.$$

The notion of a categorical quotient is often too coarse to reflect the geometry of the orbit space. The following refined notions of quotient capture more of the structure we expect from a quotient in algebraic geometry:

Definition 1.5.2. *A morphism $\varphi : X \to Y$ is a **good quotient** for the action of G on X if*

1. *φ is G-invariant.*
2. *φ is surjective.*
3. *If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X\left(\varphi^{-1}(U)\right)$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X\left(\varphi^{-1}(U)\right)^G$.*
4. *If $W \subset X$ is a G-invariant closed subset of X, its image $\varphi(W)$ is closed in Y.*
5. *If W_1 and W_2 are disjoint G-invariant closed subsets, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.*
6. *φ is affine (i.e. the preimage of every affine open is affine).*

Definition 1.5.3. *A morphism $\varphi : X \to Y$ is a **geometric quotient** for the action of G on X if it is a good quotient and the preimage of each point is a single orbit.*

Definition 1.5.4. *Assume that $\mathcal{O}(X)^G$ is finitely generated. The **affine GIT quotient** is the morphism*

$$\varphi\colon \quad X \to X/\!/G \coloneqq \operatorname{Spec} \mathcal{O}(X)^G$$

associated to the inclusion $\varphi^* : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

GIT quotients are always guaranteed to be categorical quotients, and the space $X/\!/G$ is of finite type. Furthermore, under further assumptions on G, stronger results hold.

Definition 1.5.5. *An affine algebraic group G over k is **reductive** if it is smooth and every smooth, unipotent, normal algebraic subgroup of G is trivial.*

If G is reductive $\mathcal{O}(X)^G$ is finitely generated, as proven by Nagata [Nag63]. In this case, we have the following result:

Theorem 1.5.6 ([Hos15, Theorem 4.30]). *Let G be a reductive group acting on an affine scheme X. Then the affine GIT quotient $\varphi : X \to X//G$ is a good quotient.*

This result can be strengthened when we restrict attention to the locus of stable points:

Definition 1.5.7. *A point $x \in X$ is said to be **stable** if $\dim(\mathrm{Stab}_G(x)) = 0$ and its orbit is closed in X. We denote the set of stable points by X^s.*

Then, we have the following:

Theorem 1.5.8 ([Hos15, Proposition 4.36]). *Let G be a reductive group acting on an affine scheme X, and let $\varphi : X \to X//G$ be the affine GIT quotient. Then:*

1. *The set X^s is a G-invariant open subset of X.*
2. *The image $\varphi(X^s)$ is open in $X//G$.*
3. *The restricted morphism $\varphi : X^s \to \varphi(X^s)$ is a geometric quotient.*

Whenever a GIT quotient $X//G$ is geometric, we will denote it by X/G and simply refer to it as the **quotient**.

1.5.2 Weighted projective space

One example of an affine GIT that we will work with in the next chapters is the following:

Definition 1.5.9. *Let $(a_1, \dots, a_n)$ with $a_i \in \mathbb{N}$, and define an action of $\mathbb{G}_m$ on $\mathbb{A}^n \setminus \{0\}$ by*

$$\begin{aligned} \mathbb{G}_m \times (\mathbb{A}^n \setminus \{0\}) &\longrightarrow \mathbb{A}^n \setminus \{0\} \\ (\lambda, (x_1, \dots, x_n)) &\longmapsto (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n) \end{aligned}$$

*We define the **weighted projective space** $\mathbb{P}(a_1, \dots, a_n)$[2] as the quotient $(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m$.*

In practice, working with weighted projective spaces is often simpler when the weights are well-formed:

Definition 1.5.10. *We say that a weight $(a_1, \dots, a_n)$ is **well-formed** if any $n-1$ of the a_i are coprime. That is, for all $1 \le i \le n$*

$$\gcd(a_1, \dots, \widehat{a_i}, \dots, a_n) = 1.$$

Note that $\mathbb{G}_m$ is reductive and, if the weights are well-formed, all points in $\mathbb{A}^n \setminus \{0\}$ are stable. Hence, the quotient $(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m$ is a geometric quotient.

[2] If some of the weights are repeated, we may use exponential notation for brevity. For instance, $\mathbb{P}(1, 2^3, 3)$ denotes the weighted projective space $\mathbb{P}(1, 2, 2, 2, 3)$.

A well-formed weighted projective space is always a rational variety, and therefore cannot itself be a K3 or abelian surface. However, many important examples of K3 surfaces arise as subvarieties of weighted projective spaces.

Using adjunction to compute the canonical divisor of a hypersurface X_d in a well-formed weighted projective space $\mathbb{P}(a_1, a_2, a_3, a_4)$, one deduces that X_d can only be a K3 surface if its degree satisfies

$$d = a_1 + a_2 + a_3 + a_4.$$

However, this numerical condition alone does not suffice: additional geometric conditions are required.

If we restrict ourselves to study hypersurfaces that are **quasismooth**, meaning that the affine cone over X_d in $\mathbb{A}^n$ is smooth away from the origin, then Reid showed that there are exactly 95 families of K3 surfaces arising as quasismooth hypersurfaces in weighted projective spaces [Rei80]. This classification was later extended by Iano-Fletcher to the case of codimension two, determining all quasismooth complete-intersection K3 surfaces in 4-dimensional weighted projective spaces [IF00]. Currently, thousands of examples of families of K3 surfaces in codimension ≥ 3 are known; these can be found in the Graded Ring Database [ABR02, Bro07].

Weighted projective spaces are, by construction, highly singular. Indeed, the action of $\mathbb{G}_m$ on $\mathbb{A}^n \smallsetminus \{0\}$ is often not free: there are subschemes with non-trivial stabilisers, and their images in the quotient become singular subschemes of the weighted projective space. As a result, hypersurfaces in weighted projective space frequently acquire rational double points from intersecting the singular locus of the ambient space. These singularities cannot always be detected by computing the partial derivatives of the defining equations, so one has to be extra careful when studying the singular locus of a subvariety of weighted projective space. We will see an example of this phenomenon in section 3.2.4.

Just as weighted projective spaces arise as GIT quotients by $\mathbb{G}_m$, we can construct other toric varieties using higher-dimensional tori. For example,

Definition 1.5.11. *Let M be the $2 \times n$ matrix with entries in $\mathbb{Z}$:*

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}.$$

*We define the **scroll** $\mathbb{P}(M)$ as the quotient of $(\mathbb{A}^n \smallsetminus \{0\})$ by the following action of $\mathbb{G}_m \times \mathbb{G}_m$:*

$$\begin{aligned} (\mathbb{G}_m \times \mathbb{G}_m) \times (\mathbb{A}^n \smallsetminus \{0\}) &\longrightarrow (\mathbb{A}^n \smallsetminus \{0\}) \\ ((\lambda, 1), (t_1, \ldots, t_n)) &\longmapsto (\lambda^{a_{11}} t_1, \ldots, \lambda^{a_{1n}} t_n) \\ ((1, \mu), (t_1, \ldots, t_n)) &\longmapsto (\mu^{a_{21}} t_1, \ldots, \mu^{a_{2n}} t_n) \end{aligned}$$

We will see in subsection 3.2.1 an example of a K3 surface that is a subvariety of a scroll.

The reason K3 surfaces appear as subvarieties of weighted projective spaces is that GIT quotients of projective varieties are often constructed by projectivising graded rings. In the next section, we will explain how this process works in detail.

1.5.3 Projective GIT

For a projective scheme X with an action of a group G, there is not a canonical way to produce an open subset of X which is covered by open invariant affine subsets. Instead, this depends on a choice of an equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ by a linear representation $G \to \mathrm{GL}_{n+1}$.

A projective embedding of X corresponds to the choice of an ample line bundle L on X. We will shortly see that equivariant projective embeddings are induced by an ample linearisation of the G-action on X, which is a lift of the G-action to an ample line bundle on X such that the projection to X is equivariant and the action on the fibres is linear.

In this section, we will see that, for a reductive group G acting on a projective scheme X and a choice of ample linearisation, there exists a good quotient of an open subset of semistable points in X.

Definition 1.5.12. *Let X be a projective scheme with an action of an affine algebraic group G. A **linear G-equivariant** projective embedding of X is a group homomorphism $G \to \mathrm{GL}_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$.*

We will often simply say that the G-action on $X \hookrightarrow \mathbb{P}^n$ is **linear** to mean that we have a linear G-equivariant projective embedding of X as above.

Suppose we have a linear action of a group G on a projective scheme $X \subset \mathbb{P}^n$. Then the action of G on $\mathbb{P}^n$ lifts to an action of G on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. Since the projective embedding $X \subset \mathbb{P}^n$ is G-equivariant, there is an induced action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over $X \subset \mathbb{P}^n$. More precisely, we have

$$\mathcal{O}\left(\mathbb{A}^{n+1}\right) = k\left[x_0, \ldots, x_n\right] = \bigoplus_{r \geq 0} k\left[x_0, \ldots, x_n\right]_r = \bigoplus_{r \geq 0} H^0\left(\mathbb{P}^n, \mathcal{O}_X(r)\right)$$

and if $X \subset \mathbb{P}^n$ is cut out by a homogeneous ideal $I(X) \subset k\left[x_0, \ldots, x_n\right]$, then we have that $\tilde{X} = \operatorname{Spec} R(X)$ where $R(X) = k\left[x_0, \ldots, x_n\right]/I(X)$.

The k-algebras $\mathcal{O}(\mathbb{A}^{n+1})$ and $R(X)$ are naturally graded, and the linear action of G preserves this grading. Therefore, the invariant subalgebra

$$\mathcal{O}\left(\mathbb{A}^{n+1}\right)^G = \bigoplus_{r \geq 0} k\left[x_0, \ldots, x_n\right]_r^G$$

is a graded algebra and we have the following graded ideal:

$$R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G.$$

If $R(X)^G$ is finitely generated, for instance, when G is reductive, the inclusion of finitely generated graded k-algebras $R(X)^G \hookrightarrow R(X)$ determines a rational map

$$X \to \operatorname{Proj}\left(R(X)^G\right)$$

whose indeterminacy locus is the closed subscheme of X defined by the homogeneous ideal

$$R(X)^G_+ \coloneqq \bigoplus_{r>0} R(X)^G_r.$$

Definition 1.5.13. *Let G be a group acting linearly on a projective scheme $X \subset \mathbb{P}^n$ and assume that $R(X)^G$ is finitely generated. The **nullcone** $N \subset X$ is the closed subscheme defined by the homogeneous ideal $R(X)^G_+$.*

Definition 1.5.14. *The **semistable set** $X^{ss} = X - N$ is the open subset of X given by the complement of the nullcone. By construction, the semistable set is the domain of definition of the rational map $X \to \operatorname{Proj}(R(X)^G)$.*

The induced morphism $X^{ss} \to X/\!/G \coloneqq \operatorname{Proj}\left(R(X)^G\right)$ is called the **GIT quotient** of X by the action of G. Like in the affine case, $X/\!/G$ is a categorical quotient, and we have the following theorem:

Theorem 1.5.15 ([Hos15, Theorem 5.3]). *Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then the GIT quotient $\varphi : X^{ss} \to X/\!/G$ is a good quotient. Moreover, $X/\!/G$ is a projective scheme.*

As in the affine case, we obtain a geometric quotient when we restrict to the locus of stable points:

Definition 1.5.16. *A point $x \in X$ is **stable** if it is semistable and its G-orbit is closed in X^{ss}.*

1.5.4 Working with non-reductive groups

In this thesis, we will compute the quotients of projective varieties X by the action of a constant group scheme $\underline{G}$ associated to a finite group G. To simplify notation, from now on, whenever we say that a group G acts on a variety X defined over a field k, we will mean that there is an action of the constant group scheme $\underline{G}$ over $\operatorname{Spec}(k)$ on X.

A finite group G acting on a variety X over a field of characteristic $p \geq 0$ is reductive if and only if the order of G is not divisible by p. Therefore, whenever $p \nmid |G|$, theorem 1.5.8 ensures that the GIT quotient $X/\!/G$ is geometric.

If $p \mid |G|$, however, G is no longer reductive, so theorems 1.5.6 and 1.5.8 do not apply. Nevertheless, by a classical result of Noether [Noe26], the ring $R(X)^G$ is finitely generated, so the categorical quotient $X/\!/G$ exists.

Since any finite group has dimension zero as an algebraic group, and every orbit consists of finitely many points, we deduce that all semistable points are stable. Consequently, if $X//G$ is a good quotient, then it is also geometric.

In the particular examples of quotients that we will construct, it will be easy to see that $\varphi : X \to X//G$ is a good quotient, by checking that $X//G$ satisfy all the properties of definition 1.5.2. This ultimately boils down to checking the following conditions:

1. φ is surjective.
2. $\varphi : X \to X//G$ is a morphism, or equivalently, that $X = X^{ss}$.
3. $X//G$ has the same dimension as X.
4. The degree of φ is the order of G.

These checks have been performed in all the examples that we will present in chapters 3 and 6, even if we do not mention it.

1.5.5 Projective GIT with respect to a linearisation

The technique described in section 1.5.3 allows us to describe a wide range of quotients by group actions. However, it has a drawback: if the variety X is embedded in a high-dimensional projective space, computing the ring of invariants can quickly become intractable. Fortunately, when working with an abstract projective scheme (as we will be doing in chapter 6 with $X = \mathrm{Jac}(\mathcal{C})$, where $\mathcal{C}$ is a genus two curve), there is an alternative approach.

Recall that an ample line bundle L on X (or more precisely, some power of L) determines an embedding of X into projective space. More precisely, the pair (X, L) determines a finitely generated graded k-algebra

$$R(X, L) := \bigoplus_{r \geq 0} H^0(X, rL).$$

We may choose generators $s_i \in H^0(X, r_i L)$ for $i = 0, \dots, n$, where $r_i \geq 1$. These sections define a closed immersion

$$\begin{aligned} X &\longrightarrow \mathbb{P}(r_0, \dots, r_n) \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)] \end{aligned}$$

If we replace L by mL for m sufficiently large, we may assume all generators of

$$R(X, mL) = \bigoplus_{r \geq 0} H^0(X, mrL)$$

lie in degree one.

In this case, the sections of $H^0(X, mL)$ determine a closed immersion into projective space via

$$\begin{aligned} X &\longrightarrow \mathbb{P}(H^0(X, mL)^\vee) \\ x &\longmapsto (s \mapsto s(x)) \end{aligned}$$

Now, suppose an affine algebraic group G acts on X. We would like to carry out the above construction G-equivariantly: that is, we wish to lift the G-action to the line bundle L so that the resulting projective embedding is G-equivariant and the action of G on $\mathbb{P}^n$ is linear. This idea is made precise via the notion of a linearisation.

Definition 1.5.17. *Let X be a scheme and let G be an affine algebraic group acting on X. A* ***linearisation*** *of the G-action on X is a G-equivariant line bundle $\pi : L \to X$ such that the action of G on the fibres of L is linear: for each $g \in G$ and $x \in X$, the induced map on fibres $L_x \to L_{g\cdot x}$ is linear.*

The two examples of linearisations that we will use in this thesis are the following:

- The natural action of a group $G \leq \mathrm{GL}_{n+1}$ on $\mathbb{P}^n$, inherited from its action on $\mathbb{A}^{n+1}$ via matrix multiplication, admits a natural linearisation on the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Then, if $X \hookrightarrow \mathbb{P}^n$, this induces a linearisation on $\mathcal{O}_X(1)$. In practice, this choice of linearisation will be implicit in how we describe the group action $G \times X \to X$. That is, by specifying how the generators $g \in G$ act on $X \subset \mathbb{P}^3$ via coordinates:

$$\begin{aligned} g\colon \quad X &\longrightarrow X \\ [k_1 : k_2 : k_3 : k_4] &\longmapsto \left[\sum_{j=1}^4 a_{1j}k_j : \sum_{j=1}^4 a_{2j}k_j : \sum_{j=1}^4 a_{3j}k_j : \sum_{j=1}^4 a_{4j}k_j\right] \end{aligned}$$

  we fix a representation $\rho : G \to \mathrm{GL}_4(k)$ by setting $\rho(g)$ to be the matrix with entries a_{ij}. In doing so, we implicitly fix the linearisation of $\mathcal{O}_X(1)$ induced from the representation of G on $\mathbb{A}^{n+1}$.
- Given any linearisation π and a character $\chi : G \to \mathbb{G}_m$, one can twist the linearisation by χ to obtain a new linearisation π^χ. In our setup, twisting $\mathcal{O}_X(1)$ by χ corresponds to twisting the representation ρ that we described earlier by χ.

Let G be a reductive group acting on a projective scheme X, and let L be an ample line bundle equipped with a linearisation of the G-action. Consider the graded, finitely generated k-algebra of sections of powers of L:

$$R(X, L) \coloneqq \bigoplus_{r\geq 0} H^0(X, rL).$$

Since each line bundle rL inherits a linearisation, there is an induced action of G on each $H^0(X, rL)$, and we may consider the subalgebra of G-invariant sections:

$$R(X, L)^G \coloneqq \bigoplus_{r\geq 0} H^0(X, rL)^G.$$

This subalgebra is a finitely generated k-algebra, and $\mathrm{Proj}(R(X, L)^G)$ is a projective scheme.

Let G be a reductive group acting on a projective scheme X with respect to an ample linearisation L. Then:

Definition 1.5.18. *A point $x \in X$ is **semistable** (with respect to L) if there exists an invariant section $\sigma \in H^0(X, rL)^G$ for some $r > 0$ such that $\sigma(x) \neq 0$.*

Definition 1.5.19. *A point $x \in X$ is **stable** if $\dim(G \cdot x) = \dim(G)$ and there exists an invariant section $\sigma \in H^0(X, rL)^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and the G-orbit of x is closed in the open set $X_\sigma \coloneqq \{x \in X : \sigma(x) \neq 0\}$.*

Denote by $X^{ss}(L)$ and $X^s(L)$ the open subsets of semistable and stable points in X, respectively. Then we define the **projective GIT quotient with respect to L** to be the morphism

$$\varphi\colon \quad X^{ss}(L) \longrightarrow X/\!/_L G \coloneqq \operatorname{Proj}(R(X, L)^G)$$

associated to the inclusion $R(X, L)^G \hookrightarrow R(X, L)$.

Similarly to the previous subsections, we have the following result:

Theorem 1.5.20. *Let G be a reductive group acting on a projective scheme X, and let L be an ample linearisation of this action. Then the projective GIT quotient with respect to L is a good quotient. Moreover, $X/\!/_L G$ is a projective scheme equipped with a natural ample line bundle L' such that $\varphi^* L' = nL$ for some $n > 0$.*

As before, there also exists an open subset $Y^s \subset X/\!/_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$, and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.

We emphasise that both $X^{ss}(L)$ and $X/\!/_L G$ depend on the choice of linearisation L of the G-action. As we will see in section 6.4, a smart choice of linearisation can significantly simplify the geometry of the resulting quotient.

2 The classification of K3 quotients of abelian surfaces

2.1 Introduction

As motivated in section 1.4, Kummer surfaces are among the most important examples of K3 surfaces, as we can use the properties that we know about abelian surfaces to learn about their geometry. As we saw, classical Kummer surfaces arise as the quotients of an abelian surface A by an action of the group C_2, which is generated by the involution ι.

This naturally leads to the following question:

Question 2.1.1. *Are there other finite groups G acting on an abelian surface A such that the quotient A/G is a K3 surface?*

Motivated by this, we have the following definition:

Definition 2.1.2. *Let A be an abelian surface and G a finite group acting on A. We say that the quotient A/G is a* ***generalised Kummer surface*** *if the minimal resolution of singularities of A/G is a K3 surface. We denote this quotient by $\mathrm{Kum}_G(A)$.*

In 1987, Katsura answered question 2.1.1 in the paper *Generalized Kummer surfaces and their unirationality in characteristic p*, where he proved a criterion for when such quotients exist and constructed many examples, both in characteristic zero and in positive characteristic.

In fact, Katsura classified all generalised Kummer surfaces that can arise in characteristic $p \geq 0$, except when $p = 2$, 3, or 5 [Kat87, Theorem 3.7]. More recently, Rybakov refined this classification by determining, in the positive characteristic setting, which generalised Kummer surfaces can occur over finite fields, again, excluding the cases $p = 2$, 3, and 5.

The goal of this chapter is to extend the classification to these remaining characteristics.

We say that a group G acting on an abelian surface A **preserves the group law** if the action factors through a subgroup of the group of invertible elements of $\mathrm{End}(A)^\times$.

In this setting, we have the following result:

Theorem 2.1.3. *Let A be an abelian surface defined over a field of characteristic $p \geq 0$, and let G be a finite group acting on A and preserving the group law. Furthermore, if $p \mid |G|$, assume that A is not supersingular. Then, the quotient A/G is a generalised Kummer surface if and only if G and A satisfy the conditions listed in the table below, and the action of G on A is as in Theorem 2.2.3.*

G	Conditions		Singularities of A/G
C_2	$p \neq 2$		$16A_1$
	$p = 2$	$f(A) = 2$	$4D_4^1$
		$f(A) = 1$	$2D_8^2$
C_3	$p \neq 3$		$9A_2$
	$p = 3$	$f(A) = 2$	$3E_6^1$
C_4	$p \neq 2$		$4A_3 + 6A_1$
	$p = 2$	$f(A) = 2$	$2E_7^3 + D_4^1$
C_6	$p \neq 2, 3$		$A_5 + 4A_2 + 5A_1$
	$p = 2$	$f(A) = 2$	$E_6^1 + D_4^1 + 4A_2$
	$p = 3$	$f(A) = 2$	$E_7^1 + E_6^1 + 5A_1$
Q_8	$p \neq 2$	Q_8-fixed points: 2	$2D_4 + 3A_3 + 2A_1$
		Q_8-fixed points: 4	$4D_4 + 3A_1$
Q_{12}	$p \neq 2, 3$		$D_5 + 3A_3 + 2A_2 + A_1$
	$p = 2$	$f(A) = 2$	$E_8^4 + E_7^3 + 2A_2$
$\mathrm{SL}_2(\mathbb{F}_3)$	$p \neq 2, 3$		$E_6 + D_4 + 4A_2 + A_1$
	$p = 3$	$f(A) = 2$	$E_8^2 + E_6^1 + D_4 + A_1$
C_5	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$5A_4$
C_8	$p \equiv \pm 3 \mod 8$	$f(A) = 0$	$2A_7 + A_3 + 3A_1$
C_{10}	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$A_9 + 2A_4 + 3A_1$
C_{12}	$p \equiv \pm 5 \mod 12$	$f(A) = 0$	$A_{11} + A_3 + 2A_2 + 2A_1$
Q_{16}	$p \equiv \pm 3 \mod 8$	$f(A) = 0$	$2D_6 + D_4 + A_3 + A_1$
Q_{20}	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$D_7 + A_4 + 3A_3$
Q_{24}	$p \equiv \pm 5 \mod 12$	$f(A) = 0$	$D_8 + D_4 + 2A_3 + A_2$
$\mathrm{ESL}_2(\mathbb{F}_3)$	$p \equiv \pm 3 \mod 8$	$f(A) = 0$	$E_7 + D_6 + A_3 + 2A_2$
$\mathrm{SL}_2(\mathbb{F}_5)$	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$E_8 + D_4 + A_4 + 2A_2$

Table 2.1: The classification of generalised Kummer surfaces.

As with classical Kummer surfaces, the singular points of $\mathrm{Kum}_G(A)$, contribute to the Néron–Severi lattice of its desingularisation S. This imposes strong constraints on $\mathrm{NS}(S)$, as shown by Garbagnati in [Gar17]. In the case where $G = C_3$, more is known: both the generators of the lattice and the Fourier–Mukai partners of $\mathrm{Kum}_{C_3}(A)$ have been described [RS25].

In chapters 3 and 6, we will see where the singular points come from, when we construct explicit examples of generalised Kummer surfaces arising from products of elliptic curves and from Jacobians of genus two curves.

2.2 Katsura's classification

In order to prove theorem 2.1.3, we need to determine when the quotient of an abelian surface by a group action is a generalised Kummer surface. As discussed in the introduction, most of the work has already been carried out by Katsura [Kat87] and Rybakov [Ryb24], so we will build on their results and focus on what happens in characteristics 2, 3, and 5.

Let A be an abelian surface, and let G be a finite group acting on A that preserves the group law. Let ω_A be a non-zero regular 2-form on A. This is unique up to a scalar, as $h^{2,0}(A) = 1$. Then, given an element g of order n in G, we must have $g^*\omega_A = \zeta_m \omega_A$, where $m \mid n$ and ζ_m is an m-th root of unity.

Definition 2.2.1. *The action of a finite group G on an abelian variety A is said to be **symplectic** if $g^*\omega_A = \omega_A$ for all $g \in G$.*

In the case where $A = E_1 \times E_2$ is a product of two elliptic curves, we have $\omega_A = \omega_1 \wedge \omega_2$, where ω_i is a global differential on E_i. A similar description for the case when A is the Jacobian of a genus two curve can be found in subsection 6.2.1.

Definition 2.2.2. *The action of a finite group G on an abelian variety A is said to be **rigid** if and only if every non-trivial $g \in G$ fixes a finite, non-zero number of points.*

We have now defined all the pieces that we need to state Katsura's theorem:

Theorem 2.2.3 ([Kat87, Theorem 2.4]). *Assume that the characteristic of the base field is not two. Then A/G is a generalised Kummer surface if and only if G satisfies the following conditions:*

- *The action of G on A is rigid.*
- *The action of G on A is symplectic.*
- *The quotient A/G is singular, and all the singular points are rational double points.*

If the desingularisation of A/G is a K3 surface, Katsura proved that, in every characteristic (including two), these three conditions must be satisfied. As for the converse, the proof relies on the observation that if the three conditions are satisfied, then the canonical bundle of the minimal resolution of A/G is trivial. Moreover, since the action is rigid, A/G has singular points, and thus the resolution S contains curves whose self-intersection number is negative. This implies that S cannot be an abelian, hyperelliptic, or quasi-hyperelliptic surface. If the characteristic is not equal to two, we can then deduce that S must be a K3 surface.

In characteristic two, this implication no longer holds in general, but under additional assumptions, one can still conclude that the quotient is a K3 surface.

Proposition 2.2.4. *If the characteristic of the base field is two, then theorem 2.2.3 still holds if we assume that the Picard number of the minimal resolution of A/G is greater than 10.*

Proof. Kodaira's classification of surfaces states that if the characteristic of the base field is not two, then any minimal surface S with trivial canonical bundle must be either abelian, hyperelliptic, quasi-hyperelliptic or K3. However, in characteristic two, there is the additional possibility that S is a non-classical Enriques surface defined as the quotient of a K3 surface by the action of the group schemes $\mathbb{Z}/2\mathbb{Z}$ or α_2 [Lie13, Section 7]. Any condition that allows us to discard this case will suffice to deduce that the quotient is a K3 surface, for example, if $\rho(S) > 10$, then S must be a K3 surface, as the second Betti number of an Enriques surface is at most 10. □

Showing that the Picard number of the resolution is greater than 10 is an easy condition to verify from the singularities of A/G. In particular, in all the quotients considered in this thesis, the number of exceptional divisors appearing in the resolution of the singularities of A/G forces the Picard number of the minimal resolution to be at least 17. It is important to note that the three conditions in theorem 2.2.3 are necessary. For instance, we will use the following result to deduce the rationality of A/G in several cases.

Proposition 2.2.5 ([Kat87, Theorem 2.11]). *Let G be a finite group acting on an abelian surface A such that no non-trivial element of G fixes a curve pointwise. If A/G has at least one singularity that is not a rational double point, then A/G is a rational surface.*

From theorem 2.2.3, Katsura produced a list of all finite groups G such that A/G can possibly be a generalised Kummer surface. Rybakov's refined version of the classification states the following:

Theorem 2.2.6 ([Ryb24, Proposition 5.1]). *Let A be an abelian surface over a field of characteristic $p \geq 0$. Then, A/G is a generalised Kummer surface in the following cases:*

G	Conditions		Singularities of A/G
C_2	$p \neq 2$		$16A_1$
C_3	$p \neq 3$		$9A_2$
C_4	$p \neq 2$		$4A_3 + 6A_1$
C_6	$p \neq 2, 3$		$A_5 + 4A_2 + 5A_1$
Q_8	$p \neq 2$	Q_8-fixed points: 2	$2D_4 + 3A_3 + 2A_1$
		Q_8-fixed points: 4	$4D_4 + 3A_1$
Q_{12}	$p \neq 2, 3$		$D_5 + 3A_3 + 2A_2 + A_1$
$\mathrm{SL}_2(\mathbb{F}_3)$	$p \neq 2, 3$		$E_6 + D_4 + 4A_2 + A_1$
C_5	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$5A_4$
C_8	$p \equiv \pm 3 \mod 8$	$f(A) = 0$	$2A_7 + A_3 + 3A_1$
C_{10}	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$A_9 + 2A_4 + 3A_1$
C_{12}	$p \equiv \pm 5 \mod 12$	$f(A) = 0$	$A_{11} + A_3 + 2A_2 + 2A_1$
Q_{16}	$p \equiv \pm 3 \mod 8$	$f(A) = 0$	$2D_6 + D_4 + A_3 + A_1$
Q_{20}	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$D_7 + A_4 + 3A_3$
Q_{24}	$p \equiv \pm 5 \mod 12$	$f(A) = 0$	$D_8 + D_4 + 2A_3 + A_2$
$\mathrm{ESL}_2(\mathbb{F}_3)$	$p \equiv \pm 3 \mod 8$	$f(A) = 0$	$E_7 + D_6 + A_3 + 2A_2$
$\mathrm{SL}_2(\mathbb{F}_5)$	$p \equiv \pm 2 \mod 5$	$f(A) = 0$	$E_8 + D_4 + A_4 + 2A_2$

Table 2.2: The classification of generalised Kummer surfaces when $p \nmid G$.

The only cases left to prove of theorem 2.1.3 are the ones in characteristics $p = 2, 3$ and 5, when p divides the order of G.

2.2.1 Rigid group actions

Rybakov's proved theorem 2.2.6 using the following key idea. Given a group G, the rational group algebra $\mathbb{Q}[G]$ is isomorphic to the sum of simple algebras $\mathbb{H}_V$ corresponding to irreducible representations V over $\mathbb{Q}$. From this description, we can construct another algebra, the **rigid group algebra** $\mathbb{Q}[G]^{\mathrm{rig}}$ of G as the sum over the irreducible representations without fixed points

$$\mathbb{Q}[G]^{\mathrm{rig}} = \bigoplus_{\substack{V \text{ without} \\ \text{fixed points}}} \mathbb{H}_V.$$

Then, he proved the following:

Proposition 2.2.7 ([Ryb24, Corollary 3.9]). *There exists an abelian variety with a rigid action of G in the isogeny class of an abelian variety A, if and only if there exists a homomorphism of $\mathbb{Q}$-algebras $\mathbb{Q}[G]^{\mathrm{rig}} \to \mathrm{End}^\circ(A)$.*

Using a description of Dickson of the possible subgroups G of $\mathrm{SL}_2(\bar{k})$ [Dic01], he produced the following list of groups:

Theorem 2.2.8 ([Ryb24, Theorem 4.3]). *Let G be a finite group with a rigid and symplectic action on an abelian surface over k. Then G is one of the following groups:*

- *G is a cyclic group of order $n \in \{2, 3, 4, 5, 6, 8, 10, 12\}$. Then, $\mathbb{Q}[C_n]^{\mathrm{rig}} = \mathbb{Q}(\zeta_n)$.*
- *G is a binary dihedral group Q_{4n} of order $4n$, where $2 \le n \le 6$. Then, $\mathbb{Q}[G]^{\mathrm{rig}}$ are*

G	Q_8	Q_{12}	Q_{16}	Q_{20}	Q_{24}
$\mathbb{Q}[G]^{\mathrm{rig}}$	$\mathbb{H}_{2,\infty}$	$\mathbb{H}_{3,\infty}$	$\mathbb{H}_\infty(\mathbb{Q}(\sqrt{2}))$	$\mathbb{H}_\infty(\mathbb{Q}(\sqrt{5}))$	$\mathbb{H}_\infty(\mathbb{Q}(\sqrt{3}))$

- *G is $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{ESL}_2(\mathbb{F}_3)$, or $\mathrm{SL}_2(\mathbb{F}_5)$. Then, $\mathbb{Q}[G]^{\mathrm{rig}}$ are*

G	$\mathrm{SL}_2(\mathbb{F}_3)$	$\mathrm{ESL}_2(\mathbb{F}_3)$	$\mathrm{SL}_2(\mathbb{F}_5)$
$\mathbb{Q}[G]^{\mathrm{rig}}$	$\mathbb{H}_{2,\infty}$	$\mathbb{H}_\infty(\mathbb{Q}(\sqrt{2}))$	$\mathbb{H}_\infty(\mathbb{Q}(\sqrt{5}))$

- *$p = 2$, and G is $C_3 \rtimes C_8$, or $C_3 \times Q_8$. Then, $\mathbb{Q}[G]^{\mathrm{rig}}$ are*

G	$C_3 \rtimes C_8$	$C_3 \times Q_8$
$\mathbb{Q}[G]^{\mathrm{rig}}$	$M_2(\mathbb{Q}[\zeta_4])$	$M_2(\mathbb{Q}[\zeta_3])$

- *$p = 3$, and G is $C_3 \rtimes C_8$, $C_3 \times Q_8$, or $C_3 \rtimes Q_{16}$. Then, $\mathbb{Q}[G]^{\mathrm{rig}}$ are*

G	$C_3 \rtimes C_8$	$C_3 \times Q_8$	$C_3 \rtimes Q_{16}$
$\mathbb{Q}[G]^{\mathrm{rig}}$	$M_2(\mathbb{Q}[\zeta_4])$	$M_2(\mathbb{Q}[\zeta_3])$	$M_2(\mathbb{H}_{3,\infty})$

- *$p = 5$, and G is $\mathrm{ESL}_2(\mathbb{F}_5)$, or $C_5 \rtimes C_8$. Then, $\mathbb{Q}[G]^{\mathrm{rig}}$ are*

G	$\mathrm{ESL}_2(\mathbb{F}_5)$	$C_5 \rtimes C_8$
$\mathbb{Q}[G]^{\mathrm{rig}}$	$M_2(\mathbb{H}_{5,\infty})$	$M_2(\mathbb{H}_{5,\infty})$

Here, $\mathbb{H}_{p,\infty}$ denotes the quaternion algebra over $\mathbb{Q}$ that splits at p and at the place at ∞ and $\mathbb{H}_\infty(K)$ is the quaternion algebra over a number field K that splits at the real places of K.

In particular, the possible choices for G and p, when p divides the order of G, are

- If $p = 2$, G is C_{2n} for $1 \leq n \leq 6$, Q_{4m} for $2 \leq m \leq 6$, $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{ESL}_2(\mathbb{F}_3)$, $\mathrm{SL}_2(\mathbb{F}_5)$, $C_3 \rtimes C_8$, or $C_3 \times Q_8$.
- If $p = 3$, G is C_3, C_6, Q_{12}, Q_{24}, $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{ESL}_2(\mathbb{F}_3)$, $\mathrm{SL}_2(\mathbb{F}_5)$, $C_3 \rtimes C_8$, $C_3 \times Q_8$, or $C_3 \rtimes Q_{16}$.
- If $p = 5$, G is C_5, C_{10}, $\mathrm{SL}_2(\mathbb{F}_5)$, $\mathrm{ESL}_2(\mathbb{F}_5)$, or $C_5 \rtimes C_8$.

2.2.2 Restrictions imposed by rational double points

Katsura's theorem requires that, for A/G to be a generalised Kummer surface, every singular point of A/G must be a rational double point. As we saw in section 1.2.1, the local fundamental group of a punctured neighbourhood around a rational double point is determined by the type of the singularity.

In what follows, we will show that the quotient map $A \to A/G$ imposes constraints on the possible étale covers of the singular points of A/G, and we will use this information to eliminate some of the candidate groups from the previous list.

Let P be a closed point of a normal surface X defined over an algebraically closed field, let $\mathcal{O}^h_{X,P}$ be the henselisation of $\mathcal{O}_{X,P}$ and $U = \mathrm{Spec}(\mathcal{O}^h_{X,P})$. Recall that the local fundamental group $\pi_1(U \setminus P)$ classifies finite coverings of U that are étale except above P.

Lemma 2.2.9. *Let G be a finite group acting rigidly on an abelian surface A and let $\varphi : A \to A/G$ be the quotient map. Let P be a point of A such that $\mathrm{Stab}_G(P)$ is a normal subgroup of G and let $U = \mathrm{Spec}(\mathcal{O}^h_{\varphi(P),A/G})$. Then, $\mathrm{Stab}_G(P)$ is isomorphic to a subgroup of $\pi_1(U \setminus \varphi(P))$.*

Proof. The map $\varphi : A \to A/G$ is finite, and since the action is rigid, it is étale away from the fixed points of the action of G.

Now, as $\mathrm{Stab}_G(P)$ is a normal subgroup of G, the quotient φ factors as

$$A \xrightarrow{\phi} A/\mathrm{Stab}_G(P) \xrightarrow{\psi} A/G$$

where ψ is the quotient of $A/\mathrm{Stab}_G(P)$ by the induced action of the group $G/\mathrm{Stab}_G(P)$.

Therefore, we have the sequence of ring homomorphisms

$$\mathcal{O}_{\pi(P),A/G} \to \mathcal{O}_{\phi(P),A/\mathrm{Stab}_G(P)} \to \mathcal{O}_{P,A}$$

and applying the universal property of henselisation and taking spectra, we obtain:

$$\mathrm{Spec}(\mathcal{O}^h_{P,A}) \to \mathrm{Spec}(\mathcal{O}^h_{\phi(P),A/\mathrm{Stab}_G(P)}) \to \mathrm{Spec}(\mathcal{O}^h_{\varphi(P),A/G})$$

Now, as the map ψ is given by the quotient of $A/\mathrm{Stab}_G(Q)$ by the group action induced on A by G, we deduce that G acts freely in a neighbourhood of $\phi(P)$ and the map $\mathrm{Spec}(\mathcal{O}^h_{\phi(P),A/\mathrm{Stab}_G(P)}) \to \mathrm{Spec}(\mathcal{O}^h_{\varphi(P),A/G})$ is étale. As a consequence,

$$\pi_1(U \smallsetminus \varphi(P)) = \pi_1(\mathrm{Spec}(\mathcal{O}^h_{\phi(P),A/\mathrm{Stab}_G(P)}) \smallsetminus \phi(P)).$$

Every element $g \in \mathrm{Stab}_G(P)$ fixes $\phi(P)$ and therefore induces a finite covering of U that is étale except above $\phi(P)$. Thus, we conclude that

$$\mathrm{Stab}_G(P) \leq \pi_1(\mathrm{Spec}(\mathcal{O}^h_{\phi(P),A/\mathrm{Stab}_G(P)}) \smallsetminus \phi(P)) = \pi_1(U \smallsetminus \varphi(P)). \qquad \square$$

Proposition 2.2.10. *Let A be an abelian surface over a field of characteristic p, and let G be a finite group acting rigidly and symplectically on A such that $p \mid |G|$ and the singularities of A/G are all rational double points. Then,*

- *If $p = 2$, G can only be C_2, C_4, C_6 or Q_{12}.*
- *If $p = 3$, G can only be C_3, C_6 or $\mathrm{SL}_2(\mathbb{F}_3)$.*
- *If $p = 5$, G can only be C_5.*

Proof. Since $G \subset \mathrm{End}(A)^\times$, every element $g \in G$ fixes the identity element O of A. By lemma 2.2.9, the local fundamental group of a punctured neighbourhood around the image of O in A/G must contain G as a subgroup. Thus, G must be a subgroup of one of the local fundamental groups listed in subsection 2.2.2 for rational double point singularities in characteristic p.

In characteristic 2, the only such local fundamental groups whose order is divisible by two are C_2, C_4, C_6, Q_{12} and D_m for m odd. The only groups among those listed in theorem 2.2.8 that are subgroups of any of these are precisely C_2, C_4, C_6 and Q_{12}.

In characteristic 3, the only possible local fundamental groups of order divisible by 3 are C_3, C_6, and $\mathrm{SL}_2(\mathbb{F}_3)$, all of which appear in theorem 2.2.8.

In characteristic 5, the only possibility consistent with Theorem 2.2.8 is C_5. Furthermore, in this case, Rybakov has shown that A must be supersingular [Ryb24, Corollary 5.4]. □

Remark 2.2.11. *Note that we are not claiming that all the groups listed above necessarily give rise to generalised Kummer surfaces. However, we will later see that except for C_5 in characteristic five, we can construct examples of $\mathrm{Kum}_G(A)$ in the rest of the cases.*

2.3 Computing the type of singularities

A useful point of view to study an action of a group G on an abelian variety A is to study the induced action of G on the Tate module of A. If $p \nmid |G|$, for instance, we have the following result:

Proposition 2.3.1 ([Ryb24, Proposition 3.3]). *Let G act on an abelian variety A over a field of characteristic p, and let $\ell \neq p$ and $p \nmid |G|$. The following are equivalent:*

- *The action of G on A is **rigid** which, as we saw, meant that every non-trivial element $g \in G$ has only finitely many fixed points.*
- *The representation of G in $V_\ell(A)$ is **without fixed points**, i.e., for any $g \in G$ of order n the eigenvalues of the action of g on $V_\ell(A)$ are primitive n-th roots of unity.*

Suppose now that A is an abelian surface in characteristic p that is not supersingular. Then, as we saw earlier, the p-adic Tate module $T_p(A)$ is a free $\mathbb{Z}_p$-module of rank equal to the p-rank $f(A)$. In this setting, we can still deduce a similar criterion:

Proposition 2.3.2. *Let A be an abelian surface in characteristic p that is not supersingular, and suppose G acts rigidly on A. Then the representation of G on $V_p(A)$ is without fixed points.*

Proof. Suppose that the representation of G on $V_p(A)$ was not fixed-point free. Then there would be an element $g \in G$ of order n and a primitive r-th root of unity ζ_r with $r < n$ such that ζ_r is an eigenvalue of g. In this case, the action of g^r on $V_p(A)$ would have 1 as an eigenvalue, implying that g^r fixes a non-zero vector $v \in T_p(A)$. But then, for each $j \geq 1$, the torsion point $v_j \in A[p^j]$ corresponding to the compatible system defining v would also be fixed by g^r. This shows that there would be infinitely many fixed points, and this leads to a contradiction. □

Let $g \in G$ be an element of order n acting rigidly. Whenever $p \nmid n$ or $f(A) > 0$, from the above propositions we deduce that the representation of G in $V_p(A)$ is without fixed points. Therefore, the Tate module $T_p(A)$ is a free module over the discrete valuation ring $\mathbb{Z}_p[\zeta_n]$, where ζ_n acts as g.

Proposition 2.3.3. *Let G be a finite group acting rigidly on an abelian variety A and let ℓ be a prime. If $\ell = p$, assume that $f(A) > 0$.*

1. *If $x \in A$ is a point fixed by an element $g \in G$ of order ℓ^r, then $x \in A[\ell](\bar{k})$.*
2. *If $G = C_{\ell^r}$, the set of fixed points of G is a subgroup of $A[\ell](\bar{k})$ of order $\ell^{n(\ell,r)}$, where*

$$n(\ell, r) = \frac{\log_\ell(\#A[\ell])}{(\ell-1)\ell^{r-1}}.$$

Proof. 1. As $g \in G$ has order ℓ^r, $T_\ell(A)$ is a free module over $\mathbb{Z}_\ell[\zeta_{\ell^r}]$ and if x is a fixed point of g, then $(\zeta_{\ell^r} - 1)x = 0$. Therefore, the norm $\mathrm{Norm}_{\mathbb{Q}(\zeta_{\ell^r})/\mathbb{Q}}(\zeta_{\ell^r} - 1) = \ell$ also annihilates x, so $x \in A[\ell](\bar{k})$.

2. Let $n(\ell, r)$ be the rank of $T_\ell(A)$ as a free module over $\mathbb{Z}_\ell[\zeta_{\ell^r}]$. We also know that, as $\mathbb{Z}_\ell$-module, $\mathrm{rank}_{\mathbb{Z}_\ell}(T_\ell(A)) = \log_\ell(\#A[\ell])$ and

$$\mathbb{Z}_\ell[\zeta_{\ell^r}] = \mathbb{Z}_\ell[x]/(\Phi_{\ell^r}(x)\mathbb{Z}_\ell[x])$$

where $\Phi_{\ell^r}(x)$ is the ℓ^r-th cyclotomic polynomial. As the degree of $\Phi_{\ell^r}(x)$ is $\varphi(\ell^r) = (\ell-1)\ell^{r-1}$, we deduce that

$$n(\ell, r) = \frac{\log_\ell(\#A[\ell])}{(\ell-1)\ell^{r-1}}.$$

Now, the set of fixed points of G is isomorphic to

$$T_\ell(A)/((g-1)T_\ell(A)) \cong (\mathbb{Z}_\ell[\zeta_{\ell^r}]/((\zeta_{\ell^r} - 1)\mathbb{Z}_\ell[\zeta_{\ell^r}]))^{n(\ell,r)}.$$

We know that $\mathbb{Z}_\ell[\zeta_{\ell^r}]$ is a completely ramified extension of $\mathbb{Z}_\ell$ and one can check that $(\zeta_{\ell^r} - 1)$ is a uniformiser of $\mathbb{Z}_\ell[\zeta_{\ell^r}]$, so

$$\mathbb{Z}_\ell[\zeta_{\ell^r}]/((\zeta_{\ell^r} - 1)\mathbb{Z}_\ell[\zeta_{\ell^r}]) \cong \mathbb{F}_\ell$$

and the number of fixed points is $\ell^{n(\ell,r)}$. □

One thing that is hinted in this proposition is that for an element g of order ℓ^r to act rigidly on A, $(\ell-1)\ell^{r-1}$ must divide $\log_\ell(\#A[\ell])$. This is true because if $G = \langle g \rangle$ acts rigidly, the action of G on $V_\ell(A)$ has to be semisimple with eigenvalues given by primitive ℓ^r-th roots of unity.

As a consequence, the representation of G on $V_\ell(A)$ factors through a direct sum of characters that are ℓ^r-th roots of unity. Not all of these characters are defined over $\mathbb{Q}_\ell$, but they become defined over $\mathbb{Q}_\ell(\zeta_{\ell^r})$, where ζ_{ℓ^r} is a primitive ℓ^r-th root of unity. The Galois group $\mathrm{Gal}(\mathbb{Q}_\ell(\zeta_{\ell^r})/\mathbb{Q}_\ell)$ acts on these characters, and each primitive character of order ℓ^r has a full Galois orbit of size $[\mathbb{Q}_\ell(\zeta_{\ell^r}) : \mathbb{Q}_\ell] = \varphi(\ell^r)$.

Each such orbit spans an irreducible representation of G over $\mathbb{Q}_\ell$ of dimension $\varphi(\ell^r)$, corresponding to the sum of all Galois conjugates of a given primitive character. Therefore, to write down a rigid action of G on $V_\ell(A)$, we must be able to fit copies of these irreducible representations of dimension $\varphi(\ell^r)$ into $V_\ell(A)$, which has dimension $\log_\ell(\#A[\ell])$. Hence, we obtain the divisibility condition:

$$(\ell-1)\ell^{r-1} \mid \log_\ell(\#A[\ell]).$$

For abelian surfaces, when $\ell \neq p$, $\log_\ell(\#A[\ell]) = 4$ and we therefore deduce that the possibilities are $(\ell, r) = (2,1), (2,2), (2,3), (3,1)$ and $(5,1)$.

If $\ell = p$, $\log_\ell(\#A[\ell]) = f(A)$, so if $f(A) = 2$ the only possibilities that can happen are $(\ell, r) = (2,1), (2,2)$ and $(3,1)$, and if $f(A) = 1$ the only possibility is $(2,1)$.

If G is not cyclic, we can use the information about the Sylow ℓ-subgroups of G to deduce information about the fixed points of G:

Proposition 2.3.4 ([Ryb24, Lemma 5.2]). *Under the same assumptions as in proposition 2.3.3, we also have the following:*

1. *If H is a subgroup of G such that ℓ does not divide the order of H, then the action of H on $A[\ell](\bar{k})$ only fixes O.*
2. *Let $G^{(\ell)}$ be a fixed Sylow ℓ-subgroup of G, and let n_ℓ be the number of points in A whose stabiliser is exactly $G^{(\ell)}$. Let s_ℓ be the number of conjugacy classes of Sylow ℓ-subgroups of G. Then the total number of points that are fixed by any group Sylow ℓ-subgroup is $s_\ell n_\ell$.*

Combining the results of the propositions 2.3.3 and 2.3.4, we deduce that if p divides the order of G and $f(A) > 0$, the only possible groups G acting rigidly and symplectically on an abelian surface A such that A/G has only rational double points are the following:

G	Conditions		Singularities of A/G
C_2	$p=2$	$f(A)=2$	$4D_4^1$
		$f(A)=1$	$2D_8^2$
C_3	$p=3$	$f(A)=2$	$3E_6^1$
C_4	$p=2$	$f(A)=2$	$2E_7^3 + D_4^1$
C_6	$p=2$	$f(A)=2$	$E_6^1 + D_4^1 + 4A_2$
	$p=3$	$f(A)=2$	$E_7^1 + E_6^1 + 5A_1$
Q_{12}	$p=2$	$f(A)=2$	$E_8^4 + E_7^3 + 2A_2$
$\mathrm{SL}_2(\mathbb{F}_3)$	$p=3$	$f(A)=2$	$E_8^2 + E_6^1 + D_4 + A_1$

Table 2.3: A classification of generalised Kummer surfaces when $p \mid G$ and $f(A) > 0$.

In the next chapter, we will see that all of these cases are possible even when we restrict ourselves to the case where A is the product of two elliptic curves.

2.4 About the supersingular case

Since the start of section 2.3, we have been assuming that whenever $p \mid |G|$, the abelian surfaces we considered were not supersingular. While this may appear to be a strong assumption, we propose the following conjecture:

Conjecture 2.4.1. *Let A be a supersingular abelian surface over a field of characteristic p. Then, there is no action on A of a finite group G with $p \mid |G|$ such that the minimal resolution of A/G is a smooth K3 surface.*

This conjecture is motivated by the fact that, to the best of our knowledge, no such examples have been constructed in the literature. Moreover, in all the new examples we have computed involving actions on supersingular abelian surfaces, the resulting quotients consistently exhibit elliptic singularities. We will discuss specific instances of this phenomenon in the next chapter.

In light of conjecture 2.4.1, it is particularly relevant the work of Matsumoto on what are known as **inseparable Kummer surfaces** [Mat24]. He showed that, although the quotient of a supersingular abelian surface by $\langle \iota \rangle$ is not a K3 surface, one can construct inseparable analogues that share many key properties with classical Kummer surfaces. For example, these inseparable Kummer surfaces admit coverings $Y \to X$ such that the smooth locus of Y is a group variety.

3 Generalised Kummer surfaces associated to the product of elliptic curves

3.1 Introduction

In the previous chapter, we compiled a list of possible groups G and conditions on the abelian surface A under which $\mathrm{Kum}_G(A)$ could be a generalised Kummer surface in positive characteristic. However, it is not immediately clear why such surfaces should actually exist.

In this chapter, we will show that they do exist by constructing explicit examples in the special case where A is the product of two elliptic curves. This is because, particularly when $A = E \times E$, the endomorphism ring $\mathrm{End}(A)$ admits a simple description: it is isomorphic to the matrix ring $M_2(\mathrm{End}(E))$. As a result, group actions on A can be understood as matrix groups with entries in the ring of integers of a number field or in a quaternion algebra.

Recall from the last rows of table 2.2 that there are generalised Kummer surfaces that occur only in positive characteristic, and only when A is a supersingular abelian surface. These are the cases where G is C_5, C_{10}, C_{12}, Q_{16}, Q_{24}, $\mathrm{ESL}_2(\mathbb{F}_3)$, and $\mathrm{SL}_2(\mathbb{F}_3)$. The existence of generalised Kummer surfaces for the cyclic groups in this list was proven by Katsura [Kat87, Examples 1–11], and for the remaining groups, by Rybakov [Ryb24, Lemmas 6.6 and 6.7]. The construction of these examples is somewhat involved, and for that reason, we will not discuss them in this thesis.

Instead, we will focus on the remaining cases. For cyclic quotients, we will describe a geometric method for constructing a rigid and symplectic action on a product of two elliptic curves, and explain how to explicitly compute the quotient surface under this action.

We will also provide examples of the actions of Q_8, Q_{12}, and $\mathrm{SL}_2(\mathbb{F}_3)$ on products of elliptic curves. Furthermore, by analysing the fixed points of these actions, we will describe the singularities of the corresponding quotients. However, we will not give explicit defining equations for these quotient surfaces.

3.2 Cyclic quotients

Let E be an elliptic curve given in Weierstrass form by $\mathbb{V}(f) \subset \mathbb{P}^2$, where

$$f = y^2z + a_1xyz + a_3yz^2 - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3).$$

We will now explain how to construct the quotient of the product of two elliptic curves by actions of the groups C_2, C_3, C_4, or C_6.

3.2.1 The action of order two

The Kummer surface associated with the product of two elliptic curves has been extensively studied. For the sake of completeness, we describe here how to compute a model in a similar spirit to the description given by Shioda [Shi74].

Let E_1 and E_2 be elliptic curves given in Weierstrass form by the equations

$$\begin{aligned} E_1\colon \quad & y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3, \\ E_2\colon \quad & y^2z + b_1xyz + b_3yz^2 = x^3 + b_2x^2z + b_4xz^2 + b_6z^3. \end{aligned}$$

The involution that sends any element of $E_1 \times E_2$ to its inverse is given by the map

$$\begin{aligned} \iota\colon \quad E_1 \times E_2 &\longrightarrow E_1 \times E_2 \\ ([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) &\longmapsto ([x_1 : -y_1 - a_1x_1 - a_3z_1 : z_1], [x_2 : -y_2 - b_1x_2 - b_3z_2 : z_2]). \end{aligned}$$

We now construct the Kummer surface $\operatorname{Kum}(E_1 \times E_2)$, which is the quotient of $E_1 \times E_2$ by the action of ι. We define a grading on $\mathbb{P}^2 \times \mathbb{P}^2$ by setting the multidegree of $\{x_1, y_1, z_1\}$ to be $\binom{1}{0}$ and the multidegree of $\{x_2, y_2, z_2\}$ to be $\binom{0}{1}$. It is easy to see that the functions $\{x_1, z_1, x_2, z_2\}$ are invariant by the action of ι. Moreover, the function

$$w = z_1z_2\big(2y_1y_2 + (a_1x_1 + a_3z_1)y_2 + (b_1x_2 + b_3z_2)y_1\big) \tag{3.2.1}$$

is also invariant under ι and has multidegree $\binom{2}{2}$. One can check that x_1, z_1, x_2, z_2 and w generate all invariant functions in the function field of $E_1 \times E_2$. These invariants satisfy a relation of the form

$$w^2 + wz_1z_2(a_1x_1 + a_3z_1)(b_1x_2 + b_3z_2) = z_1z_2f_6(x_1, x_2, z_1, z_2), \tag{3.2.2}$$

for some degree six polynomial $f_6 \in k[x_1, x_2, z_1, z_2]$.

Geometrically, there is another way to arrive at this model, which we will now describe, as it will connect nicely with the method used to compute the quotient by the action of C_4. Consider the map

$$\begin{aligned} \psi\colon \quad E_1 \times E_2 &\longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ ([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) &\longmapsto ([x_1 : z_1], [x_2 : z_2]), \end{aligned}$$

which assigns to a pair of points (P_1, P_2) the vertical lines ℓ_{P_1} and ℓ_{P_2} passing through them. Note that this map is not defined at the curves $\{O\} \times E_2$ and $E_1 \times \{O\}$, but one can check that it extends to a morphism.

Observe that (P_1, P_2) and $(-P_1, -P_2)$ are mapped to the same point in $\mathbb{P}^1 \times \mathbb{P}^1$ under ψ, and therefore the map descends to a morphism

$$\psi_K\colon \quad \mathrm{Kum}(E_1 \times E_2) \to \mathbb{P}^1 \times \mathbb{P}^1.$$

This is generically a 2-to-1 cover: the fibre over a generic point (ℓ_{P_1}, ℓ_{P_2}) consists of the four points $(\pm P_1, \pm P_2)$, which yield two distinct points in the Kummer surface.

Assuming that the characteristic of the base field is not two, $\mathrm{Kum}(E_1 \times E_2)$ can be described as a double cover branched along the ramification locus of ψ_K. The ramification occurs precisely when $P_1 = -P_1$ or $P_2 = -P_2$. Therefore, $\mathrm{Kum}(E_1 \times E_2)$ can be defined by the equation

$$v^2 = z_1 z_2 (x_1 - \alpha_1 z_1)(x_1 - \alpha_2 z_1)(x_1 - \alpha_3 z_1)(x_2 - \beta_1 z_2)(x_2 - \beta_2 z_2)(x_2 - \beta_3 z_2),$$

where α_i and β_j are the x-coordinates of the 2-torsion points of E_1 and E_2, respectively. This means that

$$v^2 = z_1 z_2 (x_1^3 + a_2 x_1^2 z_1 + a_4 x_1 z_1^2 + a_6 z_1^3 + \tfrac{1}{4}(a_1 x_1 + a_3 z_1)^2)(x_2^3 + b_2 x_2^2 z_2 + b_4 x_2 z_2^2 + b_6 z_2^3 + \tfrac{1}{4}(b_1 x_2 + b_3 z_2)^2)$$

This model does not work well for fields of characteristic two; however, by considering the change of variables

$$w = 2v - \tfrac{1}{2}(a_1 x_1 + a_3 z_1)(b_1 x_2 + b_3 z_2)$$

and clearing the powers of two of the denominator, we recover the equation 3.2.2, which works well in characteristic two.

Therefore, $\mathrm{Kum}(E_1 \times E_2)$ admits a model as a hypersurface X inside the scroll $\mathbb{P}(M)$ where

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

While this model works well if we want to work in affine patches of it, it is not very convenient for other tasks such as computing the singular points of the variety.

For that purpose, we can re-embed our variety inside $\mathbb{P}(1,1,1,1,2)$ by considering the image through the following map:

$$\begin{aligned} X &\longrightarrow \mathbb{P}(1,1,1,1,2) \\ [x_1 : z_1 : x_2 : z_2 : w] &\longmapsto [x_1x_2 : x_1z_2 : x_2z_1 : z_1z_2 : w] \end{aligned}$$

Then, $\mathrm{Kum}(E\times E)$ is given by the intersection of a polynomial of weighted degree two, and the image of the equation 3.2.2 inside $\mathbb{P}(1,1,1,1,2)$, which has weighted degree four.

If the characteristic of the base field is not two, this surface has sixteen singularities of type A_1 corresponding to the fixed points of $E_1 \times E_2$ by the action, which is its 2-torsion subgroup.

If the characteristic of the base field is two, then the number and type of singular points depend on the p-rank of $E_1 \times E_2$.

- If the p-rank is 2, it has four rational singularities of type D_4^1.
- If the p-rank is 1, it has two rational singularities of type D_8^2.
- If the p-rank is 0, it has one elliptic singularity of type $(19)_0$ in the sense of Wagreich [Wag70] or $A_{*,o}+A_{*,o}+A_{*,o}+A_{*,o}+A_{*,o}$ in the sense of Laufer [Lau77]. In this case, the Kummer surface associated to this surface is not a K3 surface, as a consequence of proposition 2.2.5.

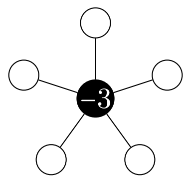

Figure 3.1: Resolution graph of the $A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o}$ singularity.

3.2.2 The action of order three

The explicit construction of $\mathrm{Kum}_{C_3}(E\times E)$ was recently described by Kondō and Mukai [KM24, Section 7.2] for elliptic curves expressed as cubics of the form

$$E: \quad x^3 + y^3 + z^3 - \lambda xyz = 0.$$

We follow similar methods to construct $\mathrm{Kum}_{C_3}(E \times E)$, but for E given in Weierstrass form. We define an automorphism of $E \times E$ of order three by

$$\begin{aligned} \tau_3 : \quad E \times E &\longrightarrow E \times E \\ (P, Q) &\longmapsto (-P - Q, P) \end{aligned}$$

Geometrically, this corresponds to the following operation: for every two points $P, Q \in E$, consider the line through them, which intersects E in a third point $-P-Q$. Then, the action τ_3 permutes cyclically the pairs of points in the line, as we can see in the diagram:

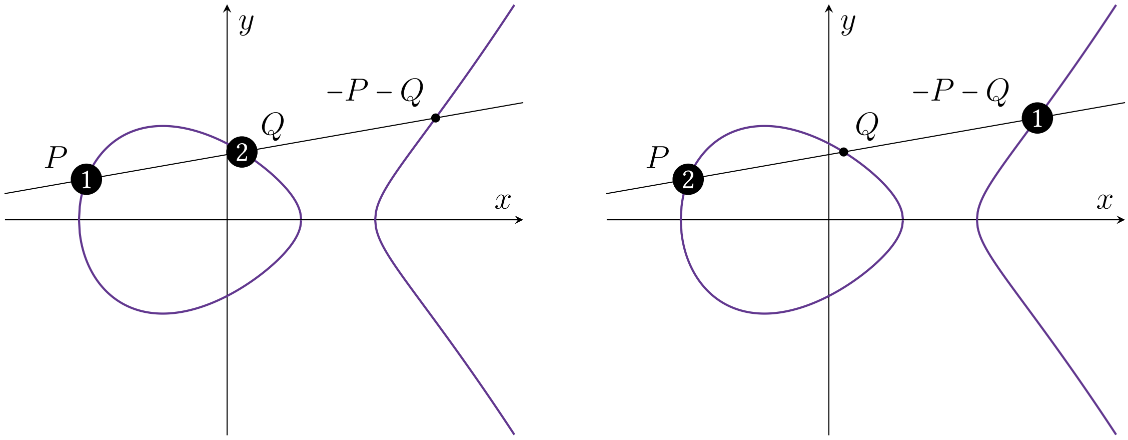

Figure 3.2: Action of τ_3 on $E \times E$.

Let $\mathrm{Kum}_{C_3}(E \times E)$ be the quotient of $E \times E$ by this action. We can find a model for $\mathrm{Kum}_{C_3}(E \times E)$ as follows. Each pair (P, Q) determines a line $\ell_{P,Q}$ in $\mathbb{P}^2$, and this line corresponds to a point in the dual projective plane $\mathbb{P}^{2\vee}$. Therefore, we can define a map

$$\begin{aligned} \psi\colon \quad E \times E &\longrightarrow \mathbb{P}^{2,\vee} \\ (P, Q) &\longmapsto \ell_{P,Q} \end{aligned}$$

As for any two points P, Q, any pair of points in the set $\{P, Q, -P-Q\}$ is sent to the same line, the map ψ factors through the quotient. Hence, we obtain an induced map $\psi_K : \mathrm{Kum}_{C_3}(E \times E) \to \mathbb{P}^2$. Generically, the preimage of a point under ψ consists of $3! = 6$ ordered pairs (P, Q), but three of these are identified in $\mathrm{Kum}_{C_3}(E \times E)$, so ψ_K is a double cover of $\mathbb{P}^2$.

To describe $\mathrm{Kum}_{C_3}(E \times E)$ explicitly, it suffices to compute the ramification locus of ψ_K. A point $(P, Q) \in E \times E$ lies in the ramification locus if and only if two of the pairs in the orbit (P, Q), $(-P-Q, P)$ and $(Q, -P-Q)$ are equal in $\mathrm{Kum}_{C_3}(E \times E)$. This happens precisely when the line $\ell_{P,Q}$ is tangent to E at one of the three points. Therefore, the ramification locus is the dual curve $E^\vee \subset \mathbb{P}^{2\vee}$, which is the image of the map

$$\begin{aligned} \phi\colon \quad E &\longrightarrow \mathbb{P}^{2,\vee} \\ P &\longmapsto \left[\frac{\partial f}{\partial x}(P) : \frac{\partial f}{\partial y}(P) : \frac{\partial f}{\partial z}(P)\right] \end{aligned}$$

This curve $E^\vee$ is a sextic, defined by the vanishing of a homogeneous degree six polynomial $h_6 \in k[u, v, w]$.

Therefore, $\mathrm{Kum}_{C_3}(E \times E)$ can be described as a double cover of $\mathbb{P}^2$ branched over $\mathbb{V}(h_6)$, and thus given by an equation of the form

$$t^2 = h_6(u, v, w) \tag{3.2.3}$$

in weighted projective space $\mathbb{P}(1, 1, 1, 3)$, assuming the characteristic is not two.

In characteristic two, this model of a curve would give rise to an inseparable cover. To construct a valid model in this case, we start in characteristic zero, find a model there that reduces well at two, and then specialise it to characteristic two. When we reduce modulo 2 the coefficients of the model given by equation 3.2.3, we obtain that $t^2 = g_3^2$ for some polynomial g_3 of degree three. Then, by lifting g_3 to characteristic zero and setting

$$f_6 = \frac{1}{4}(h_6 - g_3^2),$$

we obtain the following model of $\mathrm{Kum}_{C_3}(E \times E)$ in characteristic zero

$$t^2 + g_3(u, v, w)t + f_6(u, v, w) = 0.$$

which reduces well at two.

In characteristic different from three, $\mathrm{Kum}_{C_3}(E \times E)$ has nine A_2 singularities, arising from the fixed points of the action of C_3. These correspond to the inflection points of E, which are the 3-torsion points $E[3]$.

In characteristic three, the fixed locus of the action depends on whether E is ordinary or supersingular. If E is ordinary, then $E[3] \cong \mathbb{Z}/3\mathbb{Z}$, and $\mathrm{Kum}_{C_3}(E \times E)$ has three singularities. The resolution of these singularities reveals that the intersection matrix of the exceptional divisors is of type E_6, and the Tjurina number is seven, showing that the singularities are of type E_6^1.

If E is supersingular, then we can check that $(E \times E)/C_3$ has a single elliptic singularity, of type $D_{4,***}$ in the notation of Laufer [Lau77].

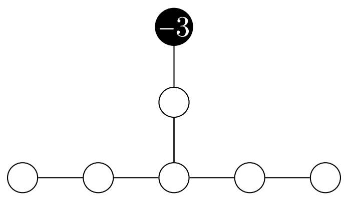

Figure 3.3: Resolution graph of the $D_{4,***}$ singularity.

In the cases where the singularities are rational double points, applying the adjunction formula, one can check that the canonical divisor of $\mathrm{Kum}_{C_3}(E \times E)$ is trivial and therefore, these are K3 surfaces. If the base field has characteristic three and E is supersingular, $(E \times E)/C_3$ is a rational surface by proposition 2.2.5.

3.2.3 The action of order four

Whenever we have that $A = E \times E$, we can define an action of order 4 on A by

$$\begin{aligned} \tau_4 : \quad E \times E &\longrightarrow E \times E \\ (P, Q) &\longmapsto (-Q, P) \end{aligned}$$

Then, $\tau_4(\tau_4(P, Q)) = (-P, -Q)$ and therefore, τ_4^2 is the involution ι.

The action can be described through the following diagram:

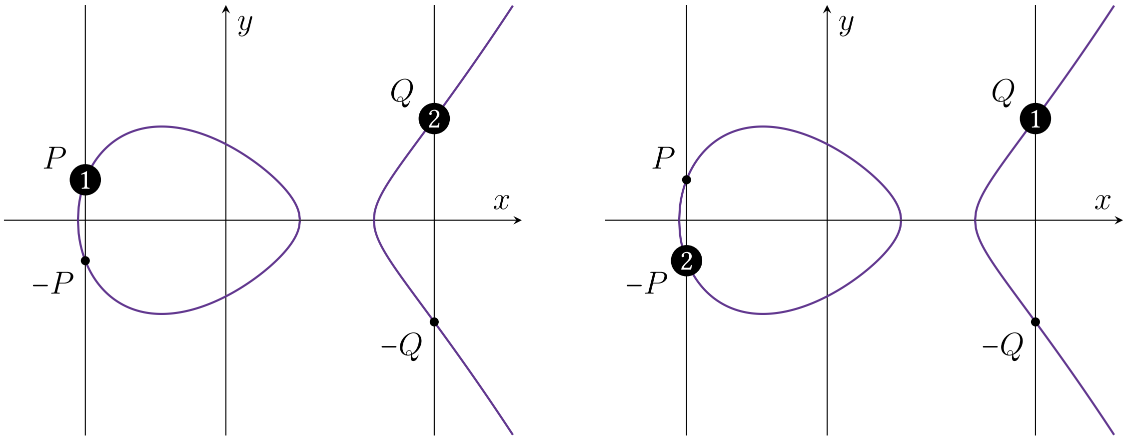

Figure 3.4: Action of τ_4 on $E \times E$.

For constructing the quotient of $E \times E$ by the action of order two, we studied the map that assigns to every ordered pair of points the corresponding ordered pair of vertical lines passing through them. For the quotient by the action of order four, we will instead consider a map ς that assigns to each pair of points the unordered pair of vertical lines passing through them.

Let $\tau_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the morphism that swaps the two copies of $\mathbb{P}^1$. The map ς is the composition of the map $\psi : E \times E \to \mathbb{P}^1 \times \mathbb{P}^1$ that we defined in subsection 3.2.1 with the quotient map $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1 \times \mathbb{P}^1)/\langle \tau_2 \rangle$. In coordinates, ς is given by

$$\begin{aligned} \varsigma : \quad E \times E &\longrightarrow \mathbb{P}^2 \\ ([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) &\longmapsto [x_1 x_2 : z_1 z_2 : x_1 z_2 + x_2 z_1] \end{aligned}$$

As with the previous case, it is straightforward to check that this map descends to a morphism on the quotient surface $\mathrm{Kum}_{C_4}(E \times E)$, and that the induced map is a double cover.

Assuming that the characteristic of the base field is not 2, the ramification locus of this map corresponds to the union of the following curves in $\mathbb{P}^2$:

1. The locus of unordered pairs of vertical lines that coincide forms a conic in $\mathbb{P}^2$.
2. The locus of unordered pairs where one of the lines is tangent to a 2-torsion point of E are four lines in $\mathbb{P}^2$, each line corresponding to a 2-torsion point.

These five curves form a highly symmetric configuration in the plane: the four lines all intersect one another, and the conic is tangent to each of the four lines, as illustrated in the diagram below:

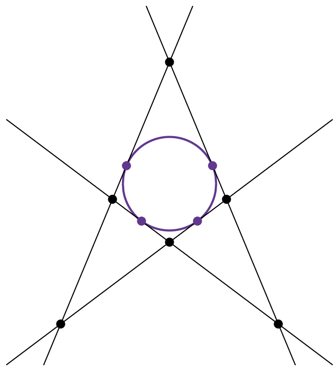

Figure 3.5: Ramification locus of the map $\operatorname{Kum}_{C_4}(E \times E) \to \mathbb{P}^2$.

This construction gives rise to a model of $\operatorname{Kum}_{C_4}(E \times E)$ as a degree six hypersurface inside $\mathbb{P}(1,1,1,3)$.

This surface has four singularities of type A_3, which are the images in the quotient of the points of the form (P,P) for some $P \in E[2]$, and six singularities of type A_1, which arise from points (P,Q) with $P,Q \in E[2]$ and $P \neq Q$. There are twelve such points, but they are identified 2-to-1 by the quotient. In the description of the surface as a double cover, all singularities correspond to singular points of the ramification locus. Specifically, the A_3 singularities are the tangency points of the conic with the lines, while the A_1 singularities correspond to the pairwise intersections of the lines.

Remark 3.2.1. *Given any configuration of four lines and a conic as described above, it is not difficult to show that, up to isomorphism, one can always construct an elliptic curve* E *such that the surface* $X_6 \subset \mathbb{P}(1,1,1,3)$*, defined as the double cover ramified along the configuration, is isomorphic to* $\operatorname{Kum}_{C_4}(E \times E)$.

To obtain a description that also works in characteristic two, we construct a set of functions that are invariant by the action of τ_4. These define the following map:

$$\begin{aligned} E \times E &\longrightarrow \mathbb{P}(1,1,1,3) \\ (P_1, P_2) &\longmapsto [x_1x_2 : z_1z_2 : x_1z_2 + x_2z_1 : (x_1z_2 - x_2z_1)v - x_2z_1^2z_2(a_1x_1 + a_3z_1)(a_1x_2 + a_3z_2)], \end{aligned}$$

where v is the function defined in equation (3.2.1).

The image of this map is a sextic hypersurface, giving a model of $\mathrm{Kum}_{C_4}(E \times E)$ that is valid in any characteristic. In particular, in characteristic two, if E is an ordinary elliptic curve, $\mathrm{Kum}_{C_4}(E \times E)$ is a K3 surface with two singularities of type E_7^3, corresponding to the images of the points (P, P) for the two distinct 2-torsion points $P \in E[2]$, and a singularity of type D_4^1 arising from the two points (P, Q) with $P \neq Q$.

If E is the unique supersingular elliptic curve in characteristic two, $\mathrm{Kum}_{C_4}(E \times E)$ is a rational surface. This can be deduced either from the fact that $\mathrm{Kum}_{C_4}(E \times E)$ is a quotient of $\mathrm{Kum}(E \times E)$, which is already rational, or from the fact that $\mathrm{Kum}_{C_4}(E \times E)$ has an elliptic singularity of type $A_{*,o} + A_{*,o} + A_{*,o} + A_{2,**,o}$, in the notation of Laufer [Lau77].

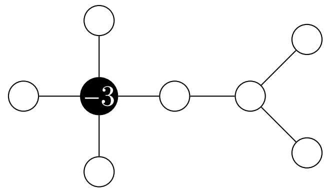

Figure 3.6: Resolution graph of the $A_{*,o} + A_{*,o} + A_{*,o} + A_{2,**,o}$ singularity.

3.2.4 The action of order six

Every elliptic curve has an action of order two given by the involution ι, and composing this with the order three action τ_3 gives rise to an action of order six in $E \times E$ by

$$\begin{aligned} \tau_6 \colon \quad E \times E &\longrightarrow E \times E \\ (P, Q) &\longmapsto (P + Q, -P). \end{aligned}$$

Geometrically, this action can be described through the following diagram:

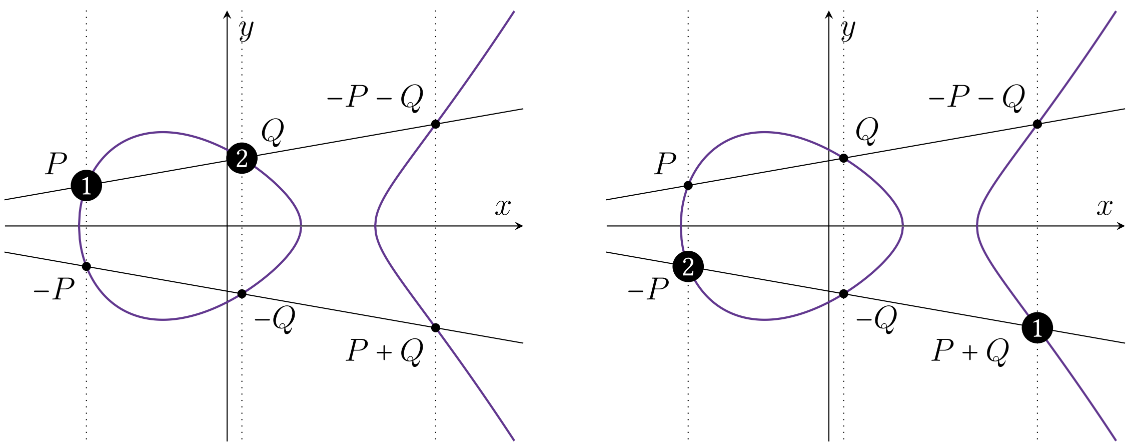

Figure 3.7: Action of τ_6 on $E \times E$.

Let $\mathrm{Kum}_{C_6}(E \times E)$ denote the quotient of $E \times E$ by this action. In a similar spirit as in the order three case, let P and Q be two different points of E. Then, the lines $\ell_{P,Q}$ and $\ell_{-P,-Q}$ can both be understood as points in $\mathbb{P}^{2,\vee}$. Now, consider the quotient of $\mathbb{P}^{2,\vee}$ by the action τ_2 that identifies $\ell_{P,Q}$ and $\ell_{-P,-Q}$. One can check that the invariant functions of the coordinate ring of $\mathbb{P}^2$ are generated by two invariants of degree one and one invariant of degree two. Therefore, $\mathbb{P}^2/\tau_2 \cong \mathbb{P}(1,1,2)$ and we can construct a map

$$\begin{aligned} \psi\colon \quad E \times E &\longrightarrow \mathbb{P}(1,1,2) \\ (P,Q) &\longmapsto [\ell_{P,Q}] \end{aligned}$$

Although the construction of this map may seem somewhat indirect, its purpose is clear. As in previous cases, it descends to the quotient, inducing a morphism

$$\psi_K\colon \quad \mathrm{Kum}_{C_6}(E \times E) \to \mathbb{P}(1,1,2)$$

Generically, the fibre over a point in $\mathbb{P}(1,1,2)$ consists of twelve pairs of points in $E \times E$ (six on each of the lines $\ell_{P,Q}$ and $\ell_{-P,-Q}$), and since six of these are identified in the quotient, we find that ψ_K is a double cover of $\mathbb{P}(1,1,2)$.

Assuming that the characteristic of the base field is not two, the ramification locus of ψ_K consists of two components:

1. The image of the dual curve $E^\vee$ by the quotient by ι, which corresponds to the vanishing of a weighted degree six polynomial.
2. The locus where $P = -Q$, i.e., where the line $\ell_{P,Q}$ is vertical. This corresponds to the vanishing of a weighted degree two polynomial in $\mathbb{P}(1,1,2)$.

Hence, the ramification divisor is cut out by a degree eight polynomial, which factors into components of degrees two and six.

As before, in characteristic two we can obtain a model by specialising from characteristic zero. This yields a surface of the form

$$t^2 + g_4(u,v,w)t + f_8(u,v,w) = 0,$$

where g_4 is a degree four polynomial (arising from the ramification divisor), and f_8 has degree eight.

If the characteristic of the base field is not two or three, then $\mathrm{Kum}_{C_6}(E \times E)$ has one A_5, four A_2 and five A_1 singularities.

As the origin is fixed by the whole group C_6, its image becomes the A_5 singularity of $\mathrm{Kum}_{C_6}(E \times E)$. There are fifteen non-trivial 2-torsion points in $E \times E$, which under the quotient contract 3-to-1 to give rise to the five A_1 in $\mathrm{Kum}_{C_6}(E \times E)$.

There are two types of orbits of this kind:

- On the one hand, we have the points of the form (P, Q) where P and Q are different 2-torsion points. In the quotient there are two points of this form, and in our model of $\mathrm{Kum}_{C_6}(E \times E)$ they correspond to the two A_1 ambient singularities that appear as a result of $\mathrm{Kum}_{C_6}(E \times E)$ being embedded in $\mathbb{P}(1, 1, 2, 4)$. These can be found as the intersection of $\mathrm{Kum}_{C_6}(E \times E)$ with the variety described by the vanishing of the two variables of degree one.
- On the other hand, we have the three points of the form (P, P) where $P \in E[2]$. As the tangent line at these is vertical, the image of these points in $\mathbb{P}(1, 1, 2, 4)$ is in the intersection of the factor of degree six and the one of degree two.

The eight non-trivial 3-torsion points fixed by τ_6^2 contract 2-to-1 to give the A_2 singularities.

If the characteristic of the base field is two and E is ordinary, $\mathrm{Kum}_{C_6}(E \times E)$ has one E_6^1, one D_4^1 and four A_2 singularities. The E_6^1 corresponds to the C_6-action at the origin. Using Magma, we can see that the intersection matrix of the exceptional curves of the desingularisation of these singularities form an E_6 configuration, and as the Tjurina number is six, we deduce that the singularity is of type E_6^1. The D_4^1 come from the fact that the three non-trivial 2-torsion points contract 3-to-1 from $\mathrm{Kum}(E \times E)$ to $\mathrm{Kum}_{C_6}(E \times E)$ and the four A_2 because the eight 3-torsion points contract 2-to-1 from $\mathrm{Kum}_{C_3}(E \times E)$ to $\mathrm{Kum}_{C_6}(E \times E)$.

If the characteristic of the base field is two and E is supersingular, we know that $(E \times E)/C_6$ is not a K3 surface, as it is a quotient of $(E \times E)/\langle\iota\rangle$, which is rational.

If the characteristic of the base field is three and E is ordinary, $\mathrm{Kum}_{C_6}(E \times E)$ has one E_7^1, one E_6^1 and five A_1 singularities. We check with Magma that the singularity at the origin is E_7^1 by resolving it and checking that the Tjurina number is seven, which rules out the possibility of it being of type E_7^0. The E_6^1 singularity is the result of the fact that the two E_6^1 singularities of $\mathrm{Kum}_{C_3}(E \times E)$ contract to one. The description of the A_1 singularities is identical to the characteristic zero case.

If the characteristic of the base field is three and E is supersingular, $(E \times E)/C_6$ still has five A_1, but instead of one E_6^1 and one E_7^1 rational double point, it has an elliptic singularity of type Tr in Laufer's notation [Lau77]. Hence, it is a rational surface.

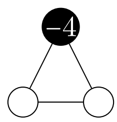

Figure 3.8: Resolution graph of the Tr singularity.

One thing that may not be apparent from the dual graph is that the exceptional lines all meet in one point, but this information can be extracted using Magma.

3.3 The quotients by Q_8, Q_{12} and $\mathrm{SL}_2(\mathbb{F}_3)$

A major difference between the cyclic cases and the quotients we are about to describe is that, whereas in the previous cases we could construct positive-dimensional families of examples, there are only finitely many quotients by the actions of Q_8, Q_{12}, and $\mathrm{SL}_2(\mathbb{F}_3)$. Many of these group actions were first described by Fujiki, who studied them in the context of automorphisms of complex tori [Fuj88].

Another important point is that none of these groups is a subgroup of $M_2(\mathbb{Z})$, since it is known that the only finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ are C_2, C_3, C_2^2, C_4, C_6, S_3, D_4, and D_6 [Vos67]. Therefore, to realise the actions by these groups, A has to be the product of curves with complex multiplication.

3.3.1 Two examples of elliptic curves that have complex multiplication

The simplest examples of elliptic curves with complex multiplication are those with j-invariants 0 and 1728, which have multiplication by ζ_3 and i, respectively. In these two cases, Fujiki showed that $E_{\zeta_3} \times E_{\zeta_3}$ and $E_i \times E_i$ both admit rigid and symplectic actions by all three groups Q_8, Q_{12}, and $\mathrm{SL}_2(\mathbb{F}_3)$ [Fuj88, Theorem 4.2].

However, neither of these curves has good ordinary reduction at the primes $p = 2$ or 3, so they cannot be used to construct examples of $\mathrm{Kum}_{Q_{12}}(A)$ in characteristic two or $\mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}(A)$ in characteristic three. For this reason, we now present two less common examples of complex multiplication elliptic curves, found in the LMFDB [The25].

The curve with complex multiplication by $\sqrt{2}i$

Let $E_{\sqrt{2}i}$ denote the unique elliptic curve (up to isomorphism) with complex multiplication by $\sqrt{2}i$. It has j-invariant 8000, and a model over $\mathbb{Z}$ is

$$y^2 = x^3 - x^2 - 3x - 1.$$

The map corresponding to multiplication by $\sqrt{2}i$ is given by the 2-isogeny:

$$\begin{aligned} [\sqrt{2}i]: \quad & E_{\sqrt{2}i} \longrightarrow E_{\sqrt{2}i} \\ & (x,y) \longmapsto \left(-\frac{x^2+1}{2(x+1)}, \frac{\sqrt{2}i(x^2+2x-1)y}{4(x+1)^2} \right) \end{aligned}$$

The curve $E_{\sqrt{2}i}$ has bad reduction only at $p = 2$, and has good ordinary reduction at $p = 3$. The reduction at three defines a curve over $\mathbb{F}_3$ whose Frobenius endomorphism has characteristic polynomial $x^2 - 2x + 3$, and thus acts as multiplication by $1 + \sqrt{2}i$.

The curve with complex multiplication by $\frac{1}{2}(1+\sqrt{7}i)$

Let E_ρ denote the unique elliptic curve (up to isomorphism) with complex multiplication by $\rho = \frac{1}{2}(1+\sqrt{7}i)$. It has j-invariant -3375, and a model over $\mathbb{Z}$ is

$$y^2 + xy = x^3 - x^2 - 2x - 1.$$

The map corresponding to multiplication by ρ is given by the 2-isogeny:

$$\begin{aligned}[\rho]:\quad & E_\rho \longrightarrow E_\rho \\ (x,y) \longmapsto & \left(\frac{(\rho-2)x^2-2\rho}{4x+\rho+2}, -\frac{((2\rho+4)x^2+(5\rho+2)x-4\rho+8)y+(3\rho-2)(x^3+x^2-2x-1)}{16x^2+(8\rho+16)x+2}\right)\end{aligned}$$

The curve E_ρ has bad reduction only at $p=7$, and good ordinary reduction at $p=2$. Its reduction at two defines a curve over $\mathbb{F}_2$ with characteristic polynomial x^2-x+2. Therefore, the Frobenius acts on this curve as multiplication by $\frac{1}{2}(1+\sqrt{7}i)$.

3.3.2 The actions by Q_8

Let $A = E_{\sqrt{2}i} \times E_{\sqrt{2}i}$. We will now construct two different actions of Q_8 on A such that the quotients by the two actions have different singular points.

Q_8 can be described by the presentation $Q_8 = \langle \sigma_4, \tau_4 \,|\, \sigma_4^4, \sigma_4^2\tau_4^2, \sigma_4\tau_4\sigma_4\tau_4^{-1}\rangle$ and, thus, we can see that Q_8 acts on A by considering the following two automorphisms:

$$\begin{aligned}\sigma_4\colon\quad E_{\sqrt{2}i}\times E_{\sqrt{2}i} &\longrightarrow E_{\sqrt{2}i}\times E_{\sqrt{2}i} \\ (P,Q) &\longmapsto ([\sqrt{2}i]P - Q, -P-[\sqrt{2}i]Q)\end{aligned}
\qquad
\begin{aligned}\tau_4\colon\quad E_{\sqrt{2}i}\times E_{\sqrt{2}i} &\longrightarrow E_{\sqrt{2}i}\times E_{\sqrt{2}i} \\ (P,Q) &\longmapsto (-Q,P)\end{aligned}$$

The quotient by this action $\mathrm{Kum}_{Q_8}(E_{\sqrt{2}i}\times E_{\sqrt{2}i})$ has two D_4, three A_3 and two A_1 singularities.

We saw in subsection 3.2.3 that the fixed points of τ_4 are the subgroup of points of the form (P,P) with $P \in E_{\sqrt{2}i}[2]$, i.e. $P \in \{O, (-1,0), (1-\sqrt{2},0), (1+\sqrt{2},0)\}$. It takes a bit more work to check that the fixed points of σ_4 are

$$\{O, ((-1,0),(-1,0)), ((1-\sqrt{2},0),(1+\sqrt{2},0)), ((1+\sqrt{2},0),(1-\sqrt{2},0))\}.$$

Therefore, two points are fixed by Q_8: O and $((-1,0),(-1,0))$. The images of these two points by the quotient map are the two D_4 singularities.

One can check that there are three subgroups of Q_8 isomorphic to C_4: $\langle\sigma_4\rangle$, $\langle\tau_4\rangle$ and $\langle\sigma_4\tau_4\rangle$. One can check that each of these groups fix O and $((-1,0),(-1,0))$ and two other different 2-torsion points.

So, in total, there are six 2-torsion points that are fixed by a unique copy of C_4, and through the quotient map, they map 2-to-1 to the three A_3 singularities. The eight remaining 2-torsion points of $E_{\sqrt{2}i} \times E_{\sqrt{2}i}$ that are not fixed by any copy of C_4 map 4-to-1 to the A_1 singularities.

A different action of Q_8 on A is the one generated by the following two automorphisms:

$$\begin{aligned} \varsigma_4\colon \quad E_{\sqrt{2}i} \times E_{\sqrt{2}i} &\longrightarrow E_{\sqrt{2}i} \times E_{\sqrt{2}i} \\ (P,Q) &\longmapsto (P + [\sqrt{2}i]Q, [\sqrt{2}i]P - Q) \end{aligned}$$

$$\begin{aligned} \upsilon_4\colon \quad E_{\sqrt{2}i} \times E_{\sqrt{2}i} &\longrightarrow E_{\sqrt{2}i} \times E_{\sqrt{2}i} \\ (P,Q) &\longmapsto ((-1 + [\sqrt{2}i])P - 2Q, -[\sqrt{2}i]P + (1 - [\sqrt{2}i])Q) \end{aligned}$$

The quotient by this action $\mathrm{Kum}_{Q_8}(E_{\sqrt{2}i} \times E_{\sqrt{2}i})$ has four D_4 and three A_1 singularities.

The fixed points of ς_4 are those of the form $(P,Q) \in (E_{\sqrt{2}i} \times E_{\sqrt{2}i})[2]$, where both P and Q are in the kernel of $[\sqrt{2}i]$, so $P, Q \in \{O, (-1,0)\}$. It can be shown that these same points fixed by υ_4, and therefore, by the whole Q_8. The image of these points by the quotient map would be the four D_4 singularities.

The other twelve 2-torsion points are not fixed by any copy of $C_4 < Q_8$ and, therefore, they map 4-to-1 to the three A_1 singularities in the quotient.

3.3.3 The action by Q_{12}

Let $A = E_\rho \times E_\rho$, and consider the automorphisms

$$\begin{aligned} \tau_4\colon \quad E_\rho \times E_\rho &\longrightarrow E_\rho \times E_\rho \\ (P,Q) &\longmapsto (P + [\rho]Q, ([\rho]-1)P - Q) \end{aligned} \qquad \begin{aligned} \tau_6\colon \quad E_\rho \times E_\rho &\longrightarrow E_\rho \times E_\rho \\ (P,Q) &\longmapsto (P+Q, -P) \end{aligned}$$

They satisfy that $\tau_6^3\tau_4^2 = \tau_4^4 = \tau_6\tau_4\tau_6\tau_4^{-1} = \mathrm{id}$ and therefore, they generate Q_{12}.

In characteristics not two or three, $\mathrm{Kum}_{Q_{12}}(E_\rho \times E_\rho)$ has one D_5, three A_3, two A_2 and one A_1 singularity.

It is easy to see that the image under the quotient map of O is the D_5 singularity, as locally this map is the quotient of $\mathbb{A}^2$ by the action of Q_{12}. One can check that the points fixed by τ_4 are those of the form (P,Q) where $Q \in E_\rho$ is in the kernel of $[\rho]$, and $P \in E_\rho$ is in the kernel of $([\rho]-1)$.

Note that $([\rho]-1)$ corresponds to multiplication by $\frac{1}{2}(-1+\sqrt{7}i)$, which is also a 2-isogeny, so

$$\langle P\rangle = \{O, \left(\tfrac{1}{8}(-5+\sqrt{7}i), \tfrac{1}{16}(5-\sqrt{7}i)\right)\}, \quad \langle Q\rangle = \{O, \left(\tfrac{1}{8}(-5-\sqrt{7}i), \tfrac{1}{16}(5+\sqrt{7}i)\right)\},$$

are both copies of $\mathbb{Z}/2\mathbb{Z}$ inside $E_\rho[2]$. Therefore, the points fixed by τ_4 are a subgroup $(\mathbb{Z}/2\mathbb{Z})^2 < (E_\rho \times E_\rho)[2]$. Now, there are two other subgroups of order four inside Q_{12}: $\langle \tau_4\tau_6^2\rangle$ and $\langle \tau_4\tau_6^4\rangle$, and each of these groups fixes a different $(\mathbb{Z}/2\mathbb{Z})^2 < (E_\rho \times E_\rho)[2]$. Hence, in total, there are nine non-trivial 2-torsion points fixed by a copy of C_4 inside Q_{12}. They are identified 3-to-1 in the quotient and their images are the three A_3 singularities.

There are six other 2-torsion points, which are permuted by the actions of τ_4 and τ_6^2, so under the quotient map, they map 6-to-1 to the A_1 singularity.

As for the two A_2 singularities, as we discussed in subsection 3.2.4, there is a subgroup of order nine of $(E_\rho \times E_\rho)[3]$ fixed by τ_6^2, and the action of τ_4 on the eight non-trivial gives rise to two orbits, whose images under the quotient map are the two A_2 singularities.

If the characteristic is two, the picture is quite similar. In that case, $\mathrm{Kum}_{Q_{12}}(E_\rho \times E_\rho)$ has one E_8^4, one E_7^3, and two A_2 singularities.

The image under the quotient map of O can be checked to be an E_8^4 singularity. This can be seen from the fact that C_6 is a normal subgroup of Q_{12}, and therefore there is a quotient map $\pi : \mathrm{Kum}_{C_6}(E_\rho \times E_\rho) \to \mathrm{Kum}_{Q_{12}}(E_\rho \times E_\rho)$. At the image of the origin, $\mathrm{Kum}_{C_6}(E_\rho \times E_\rho)$ has a singularity of type E_6^1, and π is an unramified double cover, so by the work of Artin, we deduce that the corresponding singularity of $\mathrm{Kum}_{Q_{12}}(E_\rho \times E_\rho)$ is of type E_8^4 [Art75, Case 4].

The 2-torsion of E_ρ is the kernel of Frobenius, and therefore, from the discussion in subsection 3.3.1, the four 2-torsion points of $(E_\rho \times E_\rho)[2]$ are the fixed points of τ_4. Furthermore, these points are also fixed by $\tau_4\tau_6^2$ and $\tau_4\tau_6^4$, so they are the fixed points of all copies of $C_4 < Q_{12}$. The other three non-trivial 2-torsion points are identified 3-to-1 in the quotient, yielding an E_7^3 singularity. The fact that this is the right type of singularity can be deduced from the analysis of the quotients by the group C_4 in characteristic two that we made in subsection 3.2.3.

Finally, the two A_2 singularities come from the image of 3-torsion points, in the same way as in the characteristic $p \neq 2$ case.

3.3.4 The action by $\mathrm{SL}_2(\mathbb{F}_3)$

Let $A = E_{\sqrt{2}i} \times E_{\sqrt{2}i}$, and consider the automorphisms

$$\begin{aligned} \tau_3\colon \quad E_{\sqrt{2}i} \times E_{\sqrt{2}i} &\longrightarrow E_{\sqrt{2}i} \times E_{\sqrt{2}i} & \varsigma_4\colon \quad E_{\sqrt{2}i} \times E_{\sqrt{2}i} &\longrightarrow E_{\sqrt{2}i} \times E_{\sqrt{2}i} \\ (P,Q) &\longmapsto (-P-Q,P) & (P,Q) &\longmapsto (P+[\sqrt{2}i]Q,[\sqrt{2}i]P-Q) \end{aligned}$$

Using Magma, one can check that these two automorphisms generate $\mathrm{SL}_2(\mathbb{F}_3)$. One can also check that the automorphism υ_4 that we described in subsection 3.3 is $\upsilon_4 = \tau_3\varsigma_4\tau_3^2$ and $Q_8 \cong \langle \varsigma_4, \upsilon_4\rangle$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$.

In characteristics not two or three, $\mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}(E_{\sqrt{2}i} \times E_{\sqrt{2}i})$ has one E_6, one D_4, four A_2 and one A_1 singularity.

Recall that the elements fixed by Q_8 were those inside $(E_{\sqrt{2}i} \times E_{\sqrt{2}i})[2]$ of the form $(P,Q) \in (E_{\sqrt{2}i} \times E_{\sqrt{2}i})[2]$ where $P,Q \in \{O,(-1,0)\}$. Then, the image under the quotient map of O is the E_6 singularity, and the other three points fixed by Q_8 map 3-to-1 to the D_4 singularity. The remaining twelve 2-torsion points are mapped 12-to-1 to the A_1 singularity.

Inside $\mathrm{SL}_2(\mathbb{F}_3)$, there are four subgroups of order three: $\langle\tau_3\rangle$, $\langle\varsigma_4\tau_3\rangle$, $\langle\tau_3\varsigma_4\rangle$, and $\langle\varsigma_4\tau_3\varsigma_4^3\rangle$. For each of these groups, the set of fixed points is a subgroup of order nine of $(E_{\sqrt{2}i} \times E_{\sqrt{2}i})[3]$. The action of $Q_8 = \langle\varsigma_4, \upsilon_4\rangle$ on the thirty-two non-trivial elements gives rise to four orbits, whose images under the quotient map are the four A_2 singularities.

In characteristic three, $\mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}(E_{\sqrt{2}i} \times E_{\sqrt{2}i})$ has one E_8^2, one E_6^1, one D_4 and one A_1 singularity.

The image under the quotient map of O can be checked to be an E_8^2 singularity, by studying the quotient map $\pi : \mathrm{Kum}_{Q_8}(E_{\sqrt{2}i} \times E_{\sqrt{2}i}) \to \mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}(E_{\sqrt{2}i} \times E_{\sqrt{2}i})$. At the image of the origin, $\mathrm{Kum}_{Q_8}(E_{\sqrt{2}i} \times E_{\sqrt{2}i})$ has a singularity of type D_4, and π is an unramified triple cover, so by the work of Artin, we deduce that the corresponding singularity of $\mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}(E_{\sqrt{2}i} \times E_{\sqrt{2}i})$ is of type E_8^2 [Art75, Section 5].

As before, we have a subgroup of order four of the 2-torsion that is fixed by Q_8. The three non-trivial points again map 3-to-1 to the D_4 singularity and the twelve other 2-torsion points map 12-to-1 to the A_1 singularity.

Finally, there are eight non-trivial 3-torsion points that are fixed by the order three subgroups of $\mathrm{SL}_2(\mathbb{F}_3)$. They get mapped 8-to-1 in the quotient, producing an E_6^1 singularity, as discussed in subsection 3.2.2.

4 | Kummer surfaces in characteristic two

This chapter is based on the paper *Explicit desingularisation of Kummer surfaces in characteristic two via specialisation*.

4.1 Introduction

In the previous chapter, we constructed many examples of generalised Kummer surfaces associated with products of elliptic curves. However, in some sense that we will make precise in chapter 5, the majority of principally polarised abelian surfaces are, in fact, Jacobians of genus two curves. In the next three chapters, we will be studying Jacobians of genus two curves and their associated generalised Kummer surfaces. In particular, in this chapter we will focus on the theory of (classical) Kummer surfaces arising from Jacobians of genus two curves.

As we saw, we can always find explicit equations for the Kummer surface of the Jacobian of a genus two curve as a singular quartic surface in $\mathbb{P}^3$. If the characteristic of the field of definition is different from two, this quartic has sixteen nodes corresponding to the 2-torsion points. It is well known that there is an explicit model of the desingularisation of this quartic as the intersection of three quadrics in $\mathbb{P}^5$, and this has connections with the computation of explicit equations of the Jacobian variety as the intersection of 72 quadrics inside $\mathbb{P}^{15}$ (section 4.2). For an exposition of this theory, we refer to the book of Cassels and Flynn [CF96].

The theory becomes more complicated in the case where the characteristic of the field of definition is two. Then, the 2-torsion of the abelian surface is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$, and its associated Kummer surface therefore has fewer singular points, but of higher complexity (section 4.3). As in the characteristic zero case, there is a way to construct an explicit model for the Kummer surface associated to the Jacobian of a genus two curve as a quartic in $\mathbb{P}^3$. However, following this construction does not generate a smooth model of the desingularisation of this quartic as the intersection of three quadrics in $\mathbb{P}^5$.

In principle, this could suggest that over a field of characteristic two, Jacobians of genus two curves and Kummer surfaces behave completely differently compared to how they behave over a field of any other characteristic. The main purpose of this chapter is to show that, while there are some differences, much of the already proven theory can be adapted to work over fields of characteristic two.

Theorem 4.1.1. *Given a curve of genus two over a perfect field of characteristic two, we can compute an explicit projective embedding of its Jacobian as the intersection of* 72 *quadrics in* $\mathbb{P}^{15}$*. We can also compute a projective embedding of a partial desingularisation of its associated Kummer surface as the intersection of three quadrics in* $\mathbb{P}^5$*.*

Moreover, both embeddings can be found by specialising from characteristic zero (section 4.4). Recently, Katsura and Kondō [KK23] used the theory of quadric line complexes to also describe equations for partial desingularisations of Kummer surfaces as the intersection of quadrics in $\mathbb{P}^5$.

Through different methods, we extend their results by showing that simpler models for these equations can always be computed over the field of definition of the curve. In order to prove the theorem, we study the geometry of Kummer surfaces in characteristic two through specialisation from suitable explicit models in characteristic zero (section 4.5). In characteristic zero, there are sixteen special curves called tropes going through the singular points of a Kummer surface, and we will see how the specialisation of these curves provides a natural way to study the desingularisation of Kummer surfaces in characteristic two (section 4.6).

This general theme of studying Kummer surfaces in positive characteristic from the reduction of a model in characteristic zero will play an even bigger role in section 4.7, where we construct an example of a Kummer surface with everywhere good reduction over a quadratic number field. To check that the Kummer surface has good reduction at all primes, we apply a criterion of Lazda and Skorobogatov [LS23] to study the reduction at two of an abelian surface with good reduction at all places which is defined over a quadratic field. This criterion involves studying the action of the absolute Galois group of the base field on the 2-torsion points, and sheds light on the conditions that have to be met for a smooth model of a Kummer surface to also reduce to a smooth surface modulo two.

We have written code that supports all the calculations and allows us to compute explicit equations for all the varieties that have been described. It can be found in this repository.

4.2 Models of Kummer surfaces in characteristic different from two

The theory of how to obtain explicit equations of a Kummer surface and its desingularisation over a field k of characteristic zero was first described by Grant in the case of genus two curves with a rational branch point [Gra90] and Cassels and Flynn in a more general case [CF96]. The following presentation of the theory is an adaptation of the description given by Flynn, Testa, and Van Luijk [FTvL12] to the case where we have a hyperelliptic curve described by a model of the form $y^2 + g(x)y = f(x)$.

Let k be a field of characteristic not equal to two, k^s a separable closure of k and $f(x) = \sum_{i=0}^{6} f_i x^i$ and $g(x) = \sum_{i=0}^{3} g_i x^i \in k[x]$ such that $\tilde{f}(x) = f(x) + \frac{1}{4}g(x)^2$ is a separable polynomial of degree six. We will denote by Ω the set of the six roots of $\tilde{f}$ in k^s, so that $k(\Omega)$ is the splitting field of f over k in k^s.

Let $\mathcal{C}$ be the smooth projective curve of genus two over k associated with the affine curve in $\mathbb{A}^2_{x,y}$ given by $y^2 + g(x)y = f(x)$, let $\mathrm{Jac}(\mathcal{C})$ denote the Jacobian of $\mathcal{C}$ and let $\mathrm{Jac}(\mathcal{C})[2]$ be its 2-torsion subgroup. All 2-torsion points are defined over $k(\Omega)$, so $\mathrm{Jac}(\mathcal{C})[2](k(\Omega)) = \mathrm{Jac}(\mathcal{C})[2](k^s)$. We will denote by $W \subset \mathcal{C}$ the set of Weierstrass points of $\mathcal{C}$, corresponding to the set $\{(\omega_i, -\frac{1}{2}g(\omega_i)) : \omega_i \in \Omega\}$ of points on the affine curve.

Let ι denote the automorphism of $\mathrm{Jac}(\mathcal{C})$ defined by sending every point to its inverse with respect to the group law and let $K_{\mathcal{C}}$ be the canonical divisor of $\mathcal{C}$ that is supported at the points at infinity, that is, $K_{\mathcal{C}} = (\infty_+) + (\infty_-)$, where ∞_+ and ∞_- are the two points at infinity, which may not be defined over the ground field individually. For any $w \in W$, the divisor $2(w)$ is linearly equivalent to $K_{\mathcal{C}}$ and $\sum_{w \in W}(w)$ is linearly equivalent to $3K_{\mathcal{C}}$. We let ι_h denote the hyperelliptic involution on $\mathcal{C}$ that sends (x, y) to $(x, -y - g(x))$. We then have that $\iota_h(\infty_\pm) = \infty_\mp$.

For any point P on $\mathcal{C}$ the divisor $(P) + (\iota_h(P))$ is linearly equivalent to $K_{\mathcal{C}}$, and there is a morphism $\mathcal{C} \times \mathcal{C} \to \mathrm{Jac}(\mathcal{C})$ sending (P_1, P_2) to the divisor class $(P_1) + (P_2) - K_{\mathcal{C}}$, which factors through the symmetric product of a curve with itself $\mathcal{C}^{(2)}$.

The induced map $\mathcal{C}^{(2)} \to \mathrm{Jac}(\mathcal{C})$ is birational and each non-zero element D_{ij} of $\mathrm{Jac}(\mathcal{C})[2](k^s)$ is represented by

$$D_{ij} = (w_i) - (w_j) \sim (w_j) - (w_i) \sim (w_i) + (w_j) - K_{\mathcal{C}}$$

for a unique unordered pair $\{w_i, w_j\}$ of distinct Weierstrass points.

We will denote by P_O and P_{ij} the image in X of the identity of the group law and D_{ij} respectively under the quotient map. Note that P_{ij} lies in the field $k(\omega_i + \omega_j, \omega_i\omega_j)$. In fact, the map $\mathcal{C}^{(2)} \to \mathrm{Jac}(\mathcal{C})$ is the blow-up of $\mathrm{Jac}(\mathcal{C})$ at the origin O of $\mathrm{Jac}(\mathcal{C})$. The inverse image of O is the curve on $\mathcal{C}^{(2)}$ that consists of all the pairs $\{P, \iota_h(P)\}$. We may therefore identify the function field $k(\mathrm{Jac}(\mathcal{C}))$ of $\mathrm{Jac}(\mathcal{C})$ with that of $\mathcal{C}^{(2)}$ which consists of the functions in the function field

$$k(\mathcal{C} \times \mathcal{C}) = k(x_1, x_2)[y_1, y_2]/(y_1^2 + g(x_1)y_1 - f(x_1), y_2^2 + g(x_2)y_2 - f(x_2))$$

which are invariant under the exchange of the indices. It is easy to check that for any two points P_1 and P_2 on $\mathcal{C}$ we have

$$(P_1) + (P_2) - K_{\mathcal{C}} \sim -(\iota_h(P_1) + \iota_h(P_2) - K_{\mathcal{C}})$$

and ι on $\mathrm{Jac}(\mathcal{C})$ is induced by the involution ι_h.

We can then check that the induced automorphism ι^* of $k(\mathrm{Jac}(\mathcal{C}))$ fixes x_1 and x_2, and changes y_1 and y_2 by $-y_1 - g(x_1)$ and $-y_2 - g(x_2)$ respectively. For any function $h \in k(\mathrm{Jac}(\mathcal{C}))$ we say that h is **even** or **odd** if we have that $\iota^*(h) = h$ or $\iota^*(h) = -h$ respectively. We will denote by X the **Kummer surface** of $\mathrm{Jac}(\mathcal{C})$, $X = \mathrm{Jac}(\mathcal{C})/\langle\iota\rangle$, and by Y the desingularised Kummer surface, that is, the blow-up of X at the image of the fixed points of ι.

We will denote by E_{ij} the (-2)-curve on Y above the singular point P_{ij} of X. Let $\mathcal{J}$ be the blow-up of $\mathrm{Jac}(\mathcal{C})$ in its 2-torsion points. We denote the (-1)-curve on $\mathcal{J}$ above the point $D_{ij} \in \mathrm{Jac}(\mathcal{C})[2]$ by F_{ij}. The involution ι on $\mathrm{Jac}(\mathcal{C})$ lifts to an involution on $\mathcal{J}$ such that the quotient is isomorphic to Y. Therefore, there is a morphism $\mathcal{J} \to Y$ with ramification divisor $\sum_{D_{ij} \in \mathrm{Jac}(\mathcal{C})[2]} F_{ij}$ that makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{J} \arrow[r] \arrow[d] & \mathrm{Jac}(\mathcal{C}) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

For any Weierstrass point $w \in W$ of $\mathcal{C}$ we define Θ_w to be the divisor on $\mathrm{Jac}(\mathcal{C})$ that is the image of the divisor $\mathcal{C} \times \{w\}$ on $\mathcal{C}^{(2)}$, that is, Θ_w consists of all divisor classes represented by $(P) - (w)$ for some point $P \in \mathcal{C}$. These Θ_w are known as **theta divisors** and their doubles are all linearly equivalent. We then have the following result:

Proposition 4.2.1 ([FTvL12, Proposition 3.1]). *Suppose $w \in W$ is a Weierstrass point defined over k. The linear system $|2\Theta_w|$ induces a morphism of $\mathrm{Jac}(\mathcal{C})$ to $\mathbb{P}^3_k$ that is the composition of the quotient map $\mathrm{Jac}(\mathcal{C}) \to X$ and a closed embedding of X into $\mathbb{P}^3_k$. The linear systems $|3\Theta_w|$ and $|4\Theta_w|$ induce closed embeddings of $\mathrm{Jac}(\mathcal{C})$ into $\mathbb{P}^8_k$ and $\mathbb{P}^{15}_k$ respectively.*

For any divisor D on $\mathrm{Jac}(\mathcal{C})$, let $\mathcal{L}(D) = H^0(\mathrm{Jac}(\mathcal{C}), \mathcal{O}_{\mathrm{Jac}(\mathcal{C})}(D))$ and let $\ell(D)$ be its dimension. Let Θ_+ and Θ_- be the images of the divisors $\mathcal{C} \times \{\infty_+\}$ and $\mathcal{C} \times \{\infty_-\}$, respectively, in $\mathrm{Jac}(\mathcal{C})$. Then, $\Theta_+ + \Theta_-$ is a rational divisor in $|2\Theta_w|$, so the maps induced by $|2\Theta_w|$ and $|4\Theta_w|$ can always be defined over the ground field, and the closed embeddings of X and $\mathrm{Jac}(\mathcal{C})$ are described by the elements in the bases of $\mathcal{L}(\Theta_+ + \Theta_-)$ and $\mathcal{L}(2(\Theta_+ + \Theta_-))$.

It can be checked that $\ell(\Theta_+ + \Theta_-) = 4$ and $\ell(2(\Theta_+ + \Theta_-)) = 16$. Now, it is possible to find explicit four even functions $k_1, \ldots, k_4$ and six odd functions $b_1, \ldots, b_6$ in $k(\mathrm{Jac}(\mathcal{C}))$ such that:

- The set $\{k_1, \ldots, k_4\}$ forms a basis for $\mathcal{L}(\Theta_+ + \Theta_-)$ and therefore the linear system defines an embedding $\varphi_{|\Theta_+ + \Theta_-|} : \mathrm{Jac}(\mathcal{C}) \hookrightarrow \mathbb{P}^3$, whose image is isomorphic to X.
- If we define $k_{ij} = k_i k_j$, $\{k_{11}, \ldots, k_{44}, b_1, \ldots, b_6\}$ is a basis for $\mathcal{L}(2(\Theta_+ + \Theta_-))$ and therefore defines an embedding $\varphi_{|2(\Theta_+ + \Theta_-)|} : \mathrm{Jac}(\mathcal{C}) \hookrightarrow \mathbb{P}^{15}$.

Then, the Kummer surface X is given by a quartic in $\mathbb{P}^3$ which has sixteen A_1 singularities, which are all defined over $k(\Omega)$.

In fact,

Proposition 4.2.2 ([FTvL12, Corollary 3.3]). *The quotient map* $\mathrm{Jac}(\mathcal{C}) \to X$ *is given by*

$$\begin{aligned} \mathrm{Jac}(\mathcal{C}) &\longrightarrow X \\ D &\longmapsto [k_1(D) : k_2(D) : k_3(D) : k_4(D)] \end{aligned}$$

Let $\mathrm{Sym}^2\mathcal{L}(\Theta_+ + \Theta_-)$ denote the symmetry product of $\mathcal{L}(\Theta_+ + \Theta_-)$ with itself.

Then the map

$$\begin{aligned} \mathrm{Sym}^2\mathcal{L}(\Theta_+ + \Theta_-) &\longrightarrow \mathcal{L}(2(\Theta_+ + \Theta_-)) \\ k_i * k_j &\longmapsto k_{ij} \end{aligned}$$

is injective as the $\{k_{ij}\}$ are linearly independent. We can therefore identify the space $\mathrm{Sym}^2\mathcal{L}(\Theta_+ + \Theta_-)$ with its image in $\mathcal{L}(2(\Theta_+ + \Theta_-))$. We can also find an embedding of Y, the desingularisation of X, in projective space by the following result:

Proposition 4.2.3 ([FTvL12, Proposition 3.6]). *There are direct sum decompositions*

$$\begin{aligned} \mathcal{L}(2(\Theta_+ + \Theta_-)) &= \langle \textit{even coordinates} \rangle &&\oplus \langle \textit{odd coordinates} \rangle \\ &= \mathrm{Sym}^2\mathcal{L}(\Theta_+ + \Theta_-) &&\oplus \mathcal{L}(2(\Theta_+ + \Theta_-))(-\mathrm{Jac}(\mathcal{C})[2]) \\ &= H^0(X, \varphi^*_{|\Theta_+ + \Theta_-|}\mathcal{O}_{\mathbb{P}^3}(2)) &&\oplus H^0(\mathcal{J}, \mathcal{O}_{\mathcal{J}}(2(\Theta_+ + \Theta_-) - \textstyle\sum F_{ij})) \end{aligned}$$

where $\mathcal{L}(2(\Theta_+ + \Theta_-))(-\mathrm{Jac}(\mathcal{C})[2])$ *is the subspace of* $\mathcal{L}(2(\Theta_+ + \Theta_-))$ *of sections vanishing on the* 2*-torsion points. Furthermore, the projection of* $\mathrm{Jac}(\mathcal{C}) \subset \mathbb{P}^{15}$ *away from the even coordinates determines a rational map*

$$\begin{aligned} \mathrm{Jac}(\mathcal{C}) &\dashrightarrow \mathbb{P}^5 \\ D &\longmapsto [b_1(D) : \cdots : b_6(D)] \end{aligned}$$

which induces the morphism $\mathcal{J} \to \mathbb{P}^5$ *associated to the linear system* $|4\Theta_w - \sum F_{ij}|$ *on* $\mathcal{J}$*, and factors as the quotient map* $\mathcal{J} \to Y$ *and a closed embedding* $Y \hookrightarrow \mathbb{P}^5$.

The even coordinates are the ones given by the functions $\{k_{ij}\}_{1\le i,j\le 4}$ and the odd ones the ones given by $\{b_i\}_{1\le i\le 6}$. As it was mentioned earlier, this basis defines an embedding of $\mathrm{Jac}(\mathcal{C})$ in $\mathbb{P}^{15}$ generated by 72 quadrics:

- A 20-dimensional subspace of the space generated by these quadrics is spanned by the equations of the form $k_{ij}k_{rs} = k_{ir}k_{js}$ for $1 \le i, j, r, s \le 4$.
- An additional relation between the k_{ij} comes from the fact that there is a relation between $\{k_1, \dots, k_4\}$ of degree four which defines the embedding of the Kummer surface in $\mathbb{P}^3$.
- The 21 relations arise from the fact that the space of quadrics of $\{b_1, \dots, b_6\}$ has dimension 21 and the product of two elements of $\mathcal{L}(2(\Theta_+ + \Theta_-))(-\mathrm{Jac}(\mathcal{C})[2])$ is an even function inside $\mathcal{L}(4(\Theta_+ + \Theta_-))$.

Therefore, it can be expressed as a polynomial of degree four on the k_i. From these relations, we can construct an explicit birational map $X \dashrightarrow Y$, defined outside of the singular locus of X, whose inverse $Y \dashrightarrow X$ is the blow-up of the sixteen singular points in X.

- Finally, it can be checked that there are eight relations between the elements of the form $b_i k_j$ with $1 \leq i \leq 6, 1 \leq j \leq 4$. Multiplying each of these relations by k_1, k_2, k_3 and k_4, we obtain 32 relations between the elements of the form $b_i k_{jr}$. Not all these relations are linearly independent, but they generate a 30-dimensional space.

4.2.1 Translation by a 2-torsion point

Given any non-zero element $D_{ij} \in \mathrm{Jac}(\mathcal{C})[2]$, we can define an automorphism τ_{ij} on $\mathrm{Jac}(\mathcal{C})$ by sending

$$\begin{aligned} \mathrm{Jac}(\mathcal{C}) &\longrightarrow \mathrm{Jac}(\mathcal{C}) \\ D &\longmapsto D + D_{ij} \end{aligned}$$

Then, the actions that τ_{ij} induces on $\mathcal{L}(\Theta_+ + \Theta_-)$ and $\mathcal{L}(2(\Theta_+ + \Theta_-))$ are linear [Fly93] and, as the involution ι commutes with τ_{ij}, we deduce that τ_{ij} induces a linear map in both X and Y, which is defined over the field of definition of P_{ij}, $k(\omega_i + \omega_j, \omega_i \omega_j)$. Therefore, we have an action of $\mathrm{Jac}(\mathcal{C})[2]$ on both X and Y defined over k, and over $k(\Omega)$, this is an action of $(\mathbb{Z}/2\mathbb{Z})^4$.

4.2.2 Tropes of a Kummer surface

It is classically known that the Kummer surface X contain sixteen conics known as **tropes** satisfying the following properties:

1. Every trope goes through six singular points.
2. Through every singular point there are six tropes going through it.

In the case where the Kummer surface arises from the Jacobian of a genus two curve, we have a nice combinatorial description of these tropes in terms of subsets of the Weierstrass points:

- There are six tropes of the form T_i corresponding to the partitions of the set $\{1, \dots, 6\}$ into sets of one and five elements of the form $\{\{i\}, \{j, k, l, m, n\}\}$. The trope T_i is defined to be the one going through the singular points

  $$\{O, P_{ij}, P_{ik}, P_{il}, P_{im}, P_{in}\}$$

  and in the model of X as a quartic in $\mathbb{P}^3$ that we have described, T_i can be defined over the field extension $k(\omega_i)$.
- There are ten tropes of the form T_{ijk} corresponding to the partitions of the set $\{1, \dots, 6\}$ into two subsets of three elements $\{\{i, j, k\}, \{l, m, n\}\}$.

In this case, $T_{ijk} = T_{lmn}$ and the trope T_{ijk} goes through the six singular points

$$\{P_{ij}, P_{ik}, P_{jk}, P_{lm}, P_{ln}, P_{mn}\}.$$

This trope is defined over the minimal field extension that is generated by the sums and products of $\{\omega_i, \omega_j, \omega_k, \omega_l, \omega_m, \omega_n\}$ which are invariant under the action of the permutations $(ijk)(lmn)$ and $(il)(jm)(kn)$.

Consider the subvariety $\mathcal{C} \times \{w_i\}$ inside $\mathcal{C}^{(2)}$. Then, another way of describing T_i is as the image of $\mathcal{C} \times \{w_i\}$ under the composition of the map $\mathcal{C}^{(2)} \to \operatorname{Jac}(\mathcal{C})$ and the quotient $\operatorname{Jac}(\mathcal{C}) \to X$. The rest of the tropes can be obtained as the images of any of these tropes by a suitable translation by a 2-torsion point, according to the rules:

$$\tau_{ij}(T_i) = T_j, \qquad \tau_{ij}(T_{ijk}) = T_k, \qquad \tau_{ij}(T_k) = T_{ijk}, \qquad \tau_{ij}(T_{ikl}) = T_{jkl},$$

where we are assuming that i, j, k, l are all different indices. According to how the polynomial $f(x) + \frac{1}{4}g(x)^2$ decomposes into irreducible polynomials over k, the number of tropes and singular points of X defined over k are described in the following table:

Partition	# tropes of type T_i	# tropes of type T_{ijk}	# singular points
$\{1,1,1,1,1,1\}$	6	10	16
$\{1,1,1,1,2\}$	4	4	8
$\{1,1,1,3\}$	3	1	4
$\{1,1,2,2\}$	2	2	4
$\{1,1,4\}$	2	0	2
$\{1,2,3\}$	1	1	2
$\{1,5\}$	1	0	1
$\{2,2,2\}$	0	0 or 4	4
$\{2,4\}$	0	0	2
$\{3,3\}$	0	1	1
$\{6\}$	0	0 or 1	1

Table 4.1: Number of tropes and singular points defined over the base field.

The number of tropes of each type is not too difficult to compute from the description that we have given, but there are two cases that are quite subtle:

1. The number of tropes of type T_{ijk} can be zero or four when $f(x) + \frac{1}{4}g(x)^2$ decomposes in 3 different quadrics. The number of tropes is four if and only if all quadrics split over the same quadratic number field, as in that case, assuming that the roots of the quadrics are $\{\omega_1, \omega_2\}$, $\{\omega_3, \omega_4\}$ and $\{\omega_5, \omega_6\}$, the tropes $\{T_{135}, T_{136}, T_{145}, T_{146}\}$ are defined over the field of definition of $\mathcal{C}$.
2. The number of tropes of type T_{ijk} can be zero or one when $f(x) + \frac{1}{4}g(x)^2$ is irreducible. The number of tropes is one if and only if the Galois group of the sextic is either C_6 or S_3, so that there is a partition of the roots $\{\{\omega_i, \omega_j, \omega_k\}, \{\omega_l, \omega_m, \omega_n\}\}$ preserved by the Galois group [AFJR15], and T_{ijk} is defined over the field of definition of $\mathcal{C}$.

The tropes for these special examples have been computed in `Examples.m`.

Consider the blow-up $Y \to X$. The preimage of every trope of X is a line in Y that we will denote by either $\hat{T}_i$ or $\hat{T}_{ijk}$. Then, in Y the tropes no longer intersect each other and they only intersect with the exceptional divisors E_O and E_{ij} according to the following rules:

$$E_O \cdot \hat{T}_i = 1, \quad E_O \cdot \hat{T}_{ijk} = 0, \quad E_{ij} \cdot \hat{T}_i = 1, \quad E_{ij} \cdot \hat{T}_k = 0, \quad E_{ij} \cdot \hat{T}_{ijk} = 1, \quad E_{ij} \cdot \hat{T}_{ikl} = 0.$$

The translation τ_{ij} also acts on the exceptional divisors by the rules

$$\tau_{ij}(E_O) = E_{ij}, \qquad \tau_{ij}(E_{ij}) = E_O, \qquad \tau_{ij}(E_{ik}) = E_{jk}, \qquad \tau_{ij}(E_{kl}) = E_{mn},$$

where i, j, k, l, m, n are all distinct indices. Let H be the pull-back of a hyperplane section of X under the blow-up map. Then, in $\mathrm{Pic}(Y)$, we can express the tropes in terms of H and the E_{ij} as

$$\begin{aligned} \hat{T}_i &= \tfrac{1}{2}(H - E_O - E_{ij} - E_{ik} - E_{il} - E_{im} - E_{in}), \\ \hat{T}_{ijk} &= \tfrac{1}{2}(H - E_{ij} - E_{ik} - E_{il} - E_{lm} - E_{ln} - E_{mn}). \end{aligned}$$

The Picard number of a Kummer surface is always $\rho+16$ where ρ is the Picard number of the abelian surface of which it is the quotient. It is therefore possible to prove that, for a sufficiently general Kummer surface, $\mathrm{Pic}(Y)$ is generated over $\mathbb{Z}$ by the classes of the sixteen exceptional curves, the sixteen tropes and the hyperplane section H [Keu97].

4.3 Kummer surfaces over fields of characteristic two

Let $\mathcal{C}$ be a genus two curve and now assume that the ground field is a perfect field k of characteristic two.

Recall that the 2-torsion of the Jacobian of $\mathcal{C}$ is given by

$$\mathrm{Jac}(\mathcal{C})[2](\overline{k}) \cong (\mathbb{Z}/2\mathbb{Z})^r,$$

where $0 \leq r \leq 2$ is the p-**rank**. Then, both the moduli space of curves of genus two and the moduli space of abelian surfaces are stratified in terms of the p-rank, and $\mathrm{Jac}(\mathcal{C})$ is one of the following:

- **Ordinary** (if the p-rank is 2).
- **Almost ordinary** (if the p-rank is 1).
- **Supersingular** (if the p-rank is 0).

In each of the cases, the singular points of the quotient $X = \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ have been found [Kat78] to be the following:

- In the ordinary case, X has four rational singularities of type D_4^1.
- In the almost ordinary case, $\mathrm{Jac}(\mathcal{C})/\langle\iota\rangle$ has two rational singularities of type D_8^2.
- In the supersingular case, $\mathrm{Jac}(\mathcal{C})/\langle\iota\rangle$ has one elliptic singularity of type $④_{0,1}^1$ in the sense of Wagreich [Wag70] (in which case, the Kummer surface associated to $\mathrm{Jac}(\mathcal{C})$ is not a K3 surface).

If we consider A to be an abelian surface, not necessarily the Jacobian of a genus two curve, we also have the additional possibility that A can be supersingular and superspecial, that is, it can be isomorphic to the product of two supersingular elliptic curves. In that case, as we saw in subsection 3.2.1, $A/\langle\iota\rangle$ has an elliptic double singularity of type $⑲_0$. This situation cannot happen for Kummer surfaces associated to Jacobians of curves of genus two [IKO86, Theorem 3.3].

In order to understand the resolution of singularities in these cases, Schröer observed that blowing-up the schematic image of $\mathrm{Jac}(\mathcal{C})[2]$ inside X generated a crepant partial resolution of the singularities [Sch09]. We claim that the equations for these partial resolutions can be obtained through a similar method as in characteristic zero.

Theorem 4.3.1. *Following the same notation as in section 4.2, for a general genus two curve $\mathcal{C}$ over a perfect field of characteristic two, inside the subspace of sections vanishing on the 2-torsion points with multiplicity at least two, $\mathcal{L}(2(\Theta_+ + \Theta_-))(-2\,\mathrm{Jac}(\mathcal{C})[2])$ there is a subspace of dimension six which generates an embedding of a surface in $\mathbb{P}^5$ as the complete intersection of three quadrics. This surface Y is a partial desingularisation of the quartic model of a Kummer surface and has the following singularities:*

- *If $\mathrm{Jac}(\mathcal{C})$ is ordinary, Y has twelve singularities of type A_1.*
- *If $\mathrm{Jac}(\mathcal{C})$ is almost ordinary, Y has two singularities of type D_4^0 and two singularities of type A_3.*
- *If $\mathrm{Jac}(\mathcal{C})$ is supersingular, Y has an elliptic singularity, which, in Laufer's notation* [Lau77, Table 3]*, has type $A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o}$.*

Furthermore, this embedding can be defined explicitly over the field of definition of the curve, and it can also be found by specialising from characteristic zero.

As mentioned in the introduction, Katsura and Kondō used the theory of line complexes to obtain similar results for Kummer surfaces that do not necessarily come from the Jacobians of genus two curves [KK23].

Furthermore, for Kummer surfaces coming from the Jacobians of ordinary genus two curves, they proved that $\mathcal{L}(2(\Theta_+ + \Theta_-))(-2\,\mathrm{Jac}(\mathcal{C})[2])$ has exactly dimension six.

The advantages of the method described in this chapter are that the scheme models that have been computed are defined over the field of definition of the curve, which is not always the case for the models of Katsura and Kondō, they have simpler equations, and also work for Jacobians of supersingular genus two curves. In section 4.5, we will provide the changes of coordinates that connect these scheme models with Katsura and Kondō's.

The proof of this theorem will be constructive, as given a genus two curve in characteristic two, we will compute the equation of its Jacobian, its corresponding Kummer surface and models for its partial desingularisations.

4.4 Computing models of Jacobian and Kummer surfaces

The equations of these surfaces will be computed through the following steps:

1. We will first compute a basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ for a general genus two curve.
2. Then, we will compute the relations between the elements of this basis to obtain the quadratic relations that the elements of the basis satisfy.
3. Finally, we will argue how these can be used to study the corresponding Kummer surfaces.

The software that have been used to perform these computations were Mathematica [WR24] for computing the majority of equations and Magma [BCP97] to perform the more geometric operations such as the blow-ups.

The code in Mathematica is classified in three notebooks: `Part 1`, `Part 2` and `Part 3` roughly computing the three steps described above. The Magma code is divided in two notebooks, one named `Functions.m` implementing the scheme models of the surfaces and another one named `Examples.m` with examples of use. All the relevant code is available here.

4.4.1 Computing a basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$

The idea of finding explicit models of Kummer surfaces in characteristic two from specialisation from the characteristic zero goes back to the work of Müller [Mül10] who, for a general genus two curve given by the equation

$$y^2 + (\sum_{i=0}^{3} g_i x^i)y = \sum_{i=0}^{6} f_i x^i,$$

computed a basis $\{k_1, k_2, k_3, k_4\}$ of $\mathcal{L}(\Theta_+ + \Theta_-)$ in characteristic zero.

From now on, we will assume to be working with genus two curves of the form

$$y^2 + g(x)y = f(x),$$

where $\deg(g) = 3$ and $\deg(f) \leq 6$, for reasons which will become apparent in the next section.

From Müller's article, we know the equations for a basis $\{k_1, k_2, k_3, k_4\}$ of $\mathcal{L}(\Theta_+ + \Theta_-)$ in characteristic zero for a general genus two curve which, when we reduce the coefficients modulo two, forms a basis $\{\overline{k}_1, \overline{k}_2, \overline{k}_3, \overline{k}_4\}$ of $\mathcal{L}(\Theta_+ + \Theta_-)$ for a general genus two curve defined over a field of characteristic two (A.1.1). As these are linearly independent, and the product of any two elements of $\mathcal{L}(\Theta_+ + \Theta_-)$ lies in $\mathcal{L}(2(\Theta_+ + \Theta_-))$, we can obtain ten elements of the basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$, which in analogy of the characteristic zero case, we will denote by $\overline{k}_{ij}$.

As $\ell(2(\Theta_+ + \Theta_-)) = 16$, we still need to compute six more independent elements of the basis, for which we will specialise from characteristic zero.

There is a small issue, which is that it is not known what the elements of these basis are for models of curves of the form $y^2 + g(x)y = f(x)$. However, for genus two curves defined by equations of the form $y^2 = \sum_{i=0}^{6} \tilde{f}_i x^i$ we know equations for a basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$, and, more specifically, for a basis $\{b_1, \ldots, b_6\}$ that generates all odd functions [FTvL12, section 3].

By considering the morphism $(x, y) \mapsto (x, y + \frac{1}{2}g(x))$, any curve $\mathcal{C}$ of the form $y^2 + g(x)y = f(x)$ can be mapped over k to a curve $\tilde{\mathcal{C}}$ of the form $y^2 = \tilde{f}(x)$ where the polynomial $\tilde{f}(x) = f(x) + \frac{1}{4}g(x)^2$. Through this change of coordinates, we can find a basis $\{b_1, \ldots, b_6\}$ for the odd functions of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ for models of curves of the form $y^2 + g(x)y = f(x)$ in characteristic zero.

One would hope that the reduction of these b_i modulo two would give us a basis of the odd functions of the reduction modulo two of the curve. That is not the case. However, we can easily construct a basis that reduces well modulo two via the following procedure (which amounts to compute the Smith normal form associated with the basis):

1. We first multiply each element of the basis by the smallest power of two that will allow us to clear all powers of two of the denominator.
2. Then, we can reduce the coefficients of these elements modulo two to obtain a new set of elements. As some of these elements are linearly dependent, we compute all linear relations among these by computing the kernel of the matrix associated with this basis over the reduced field. Lifting these linear relations to k, we obtain new elements in the basis that reduce to zero when reducing modulo two.
3. Dividing by the appropriate powers of two, we obtain new elements in the basis that reduce modulo two to elements that were not previously in the basis.

We can continue this process until we obtain a basis of odd functions whose reductions are linearly independent and belong to $\mathcal{L}(2(\Theta_+ + \Theta_-))$ (A.1.2).

However, there is an additional problem that comes from working in characteristic two, which is the fact that the reductions $\overline{b}_i$ of the newly found b_i are all linearly dependent on the $\overline{k}_{jr}$ that we have previously computed (A.1.3).

There is an intuitive reason for why this is the case, which is the following: the eigenvalues of the action of the involution ι on the elements in $\mathcal{L}(2(\Theta_+ + \Theta_-))$ are all either 1 or -1, and the basis that we have chosen is the diagonalised basis with respect to these basis. When we reduce the elements of this basis modulo two, we see that the action of ι in the elements of these basis is trivial, which we know that it cannot possibly be the case, as we can construct elements in $\mathcal{L}(2(\Theta_+ + \Theta_-))$ that are not invariant under this action.

This does not imply that constructing these $\overline{b}_i$ has been in vain. As a matter of fact, these $\overline{b}_i$, allow us to describe partial desingularisations of Kummer surfaces in characteristic two, which will be explored in the next section in detail. In addition to this, we can also construct elements of the basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ using these b_i. As the $\overline{b}_i$ can be expressed as a linear combination of elements of $\overline{k}_{jr}$, lifting these linear combinations to characteristic zero gives rise to elements of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ that reduce to zero in characteristic two, and therefore, must divide a power of two.

Following the Smith normal form procedure that has been previously described, we can construct a basis $\{v_1, \ldots, v_6\}$ such that their reduction modulo two, $\overline{v}_i$, together with the $\overline{k}_{jr}$ generate the basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ that we are looking for. This process and the resulting equations are computed in the notebook `Part 1`.

4.4.2 Computing the equations of the Jacobian

We now need to compute the equations defining the embedding of the Jacobian in projective space. That is, we need to find the 72 quadratic relations that exist among the elements of $\mathcal{L}(2(\Theta_+ + \Theta_-)) = \{\overline{v}_1, \ldots, \overline{v}_6, \overline{k}_{11}, \overline{k}_{12}, \ldots, \overline{k}_{44}\}$.

Again, we will compute these from specialisation from the characteristic zero case, from the elements $\{v_1, \ldots, v_6, k_{11}, \ldots, k_{44}\}$. The key to this is to first compute the relations in the basis that diagonalises the involution, $\{b_1, \ldots, b_6, k_{11}, \ldots, k_{44}\}$, as here working with odd and even functions greatly simplifies the process. As described before, there are twenty relations of the form

$$k_{ij}k_{rs} - k_{ir}k_{js} = 0,$$

which are easy to compute. In order to compute the rest of the relations, we adapted a strategy that Flynn [Fly90] originally used to compute these relations. Flynn observed that it was possible to define two independent weight functions on x, y and the f_i such that the equation of $\mathcal{C}$ has homogeneous weight. As a consequence of this, all existing relations between the elements of a basis must also have homogeneous weights.

Because there is only a limited amount of monomials of a certain weight, this highly restricts the possible monomials involved in a relation. We will extend this idea by defining two weight functions w_1 and w_2 on x, y, f_i and g_j by

	x	y	f_i	g_j
w_1	0	1	2	1
w_2	1	3	$6-i$	$3-j$

Table 4.2: Values of the weight functions for x, y, f_i and g_j.

From those weights, we can easily check that the weights of the elements of the basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ are the following (note that the weight of the k_{jr} are the sum of the weights of k_j and k_r):

	k_1	k_2	k_3	k_4	v_1	v_2	v_3	v_4	v_5	v_6
w_1	0	0	0	2	1	1	1	2	3	5
w_2	0	1	2	4	2	3	4	5	6	7

Table 4.3: Values of the weight functions for k_i and b_j.

We are looking for homogeneous relations between the elements of the basis. To avoid instead searching relations between rational functions, we will multiply all the k_{jr} by $(x_1 - x_2)^2$ and all the v_i by $(x_1 - x_2)^4$, so that all the functions are polynomials.

We know from the description of the relations that was described in the previous sections that there are 21 relations of the form $b_i b_j = \{\text{a quadratic polynomial on the } k_{sr}\}$. We start by computing the weights w_1 and w_2 corresponding to the product of $b_i b_j$ and we then compute all possible monomials on the variables f_i, g_j and k_{rs} of that weight. For example,

$$w_1(b_1^2) = 2, \qquad w_2(b_1^2) = 4,$$

and the only monomials with those weights are

$$\{g_1^2k_{11}^2, g_0g_2k_{11}^2, f_2k_{11}^2, g_1g_2k_{11}k_{12}, g_0g_3k_{11}k_{12}, f_3k_{11}k_{12}, g_2^2k_{11}k_{13}, g_1g_3k_{11}k_{13},\\ f_4k_{11}k_{13}, k_{11}k_{14}, g_2^2k_{11}k_{22}, g_1g_3k_{11}k_{22}, f_4k_{11}k_{22}, g_2g_3k_{11}k_{23}, f_5k_{11}k_{23},\\ g_3^2k_{11}k_{33}, f_6k_{11}k_{33}, g_2g_3k_{12}k_{22}, f_5k_{12}k_{22}, g_3^2k_{12}k_{23}, f_6k_{12}k_{23}, g_3^2k_{22}^2, f_6k_{22}^2\}.$$

We therefore deduce that a $\mathbb{Q}$-linear combination of these elements must be equal to b_1^2. In order to compute this $\mathbb{Q}$-linear combination, we could expand the expressions of the k_i in terms of x_1, x_2, y_1 and y_2 and find what this linear combination would have to be. This works for the products of b_1, b_2 and b_3 as their weights are small and there are not that many monomials with those weights.

For instance, for b_1^2, we find that the relation that we are looking for is:

$$\begin{aligned}
&b_1^2 - 4f_2k_{11}^2 - g_1^2k_{11}^2 - 4f_3k_{11}k_{12} - 2g_1g_2k_{11}k_{12} - 4f_4k_{11}k_{22} - g_2^2k_{11}k_{22} \\
&\quad - 2g_1g_3k_{11}k_{22} - 4f_5k_{12}k_{22} - 2g_2g_3k_{12}k_{22} - 4f_6k_{22}^2 - g_3^2k_{22}^2 + 2g_1g_3k_{11}k_{13} \\
&\quad + 4f_5k_{11}k_{23} + 2g_2g_3k_{11}k_{23} + 8f_6k_{12}k_{23} + 2g_3^2k_{12}k_{23} - 4f_6k_{11}k_{33} - g_3^2k_{11}k_{33} - 4k_{11}k_{14} = 0.
\end{aligned}$$

However, when we consider products involving b_4, b_5 and b_6 this approach becomes unfeasible, as there are many more possible monomials with those weights. For instance, the number of monomials of the same weight as b_6^2 is 8374. Therefore, a more efficient approach is needed to compute the $\mathbb{Q}$-linear combination that exists between the elements of a basis.

The idea behind the algorithm that we have used to compute this is the following. We are looking for a $\mathbb{Q}$-linear relation among elements that are in $\mathcal{L}(2(\Theta_+ + \Theta_-))$ so, in particular, if we pick a random curve and two random points in that curve, and we evaluate the values of the f_i, the g_j, the b_i and the k_{jr}, they should satisfy that $\mathbb{Q}$-linear relation. If we only evaluate at one curve and two points, we will only get a vector in $\mathbb{Q}^{\#\text{monomials of that weight}}$, so it will satisfy many other linear relations. Nevertheless, by evaluating in many other curves and points, we can generate more vectors satisfying these linear relations, and by generating enough vectors randomly, we can construct a matrix for which the only elements in the kernel are the $\mathbb{Q}$-linear relations we are looking for.

An important question in this algorithm is how to generate random curves and random points that will have small coefficients. The method that we have used consists in choosing random small integer values for $g_1, \dots, g_3, f_1, \dots, f_6$, for instance, in the interval $[-4, 4]$. Then, to generate two points (x_1, y_1) and (x_2, y_2) in the curve

$$y^2 + \Big(\sum_{i=0}^{3} g_i x^i\Big) y = \sum_{i=0}^{6} f_i x^i,$$

we pick two random values for x_1 and y_1, and we pick x_2 randomly and y_2 to be y_1 plus either 1 or -1. Then, we set g_0 and f_0 to be

$$\begin{aligned}
f_0 &= \frac{(y_2^2 + (\sum_{j=1}^3 g_i x_2^i) y_2 - \sum_{i=1}^6 f_i x_2^i) y_1 - (y_1^2 + (\sum_{j=1}^3 g_i x_1^i) y_1 - \sum_{i=1}^6 f_i x_1^i) y_2}{y_1 - y_2}, \\
g_0 &= \frac{(y_2^2 + (\sum_{j=1}^3 g_i x_2^i) - \sum_{i=1}^6 f_i x_2^i) - (y_1^2 + (\sum_{j=1}^3 g_i x_1^i) y_1 - \sum_{i=1}^6 f_i x_1^i)}{y_1 - y_2}.
\end{aligned}$$

By definition, g_0 and f_0 are integers (as $|y_1 - y_2| = 1$ by the choice of y_2), so we can successfully force (x_1, y_1) and (x_2, y_2) to be in the curve and, therefore, generate random curves and random points defined over the integers and with relatively small coefficients. If we generate enough of these (usually 10% more than the length of the vector suffices) and we compute the kernel of the matrix that we form with them, we obtain all the relations of the form $b_i b_j = \{\text{a quadratic polynomial on the } k_{sr}\}$.

By considering the monomials that have degree $w_1 = 4$ and degree $w_2 = 10$, we can also recover the relation defining the Kummer. So far, we have computed 42 out of the 72 relations defining the equations of the Jacobian in $\mathbb{P}^{15}$. The only ones that are left are the 30 relations only involving monomials of the form $k_{ij}b_s$ for $1 \le i, j \le 4$ and $1 \le s \le 6$. In order to find these, what we can do is to find the eight relations that exist between the elements of the form $k_i b_s$, multiply each of these relations by k_1, k_2, k_3 and k_4 to obtain 32 new relations, and then, remove the two that are a linearly combination of the rest.

With this, we obtain a set of 72 equations determining a model of the Jacobian that is valid in any characteristic different from two. The only step that we need to take to find the relations in characteristic two, is to express these relations in terms of the $\{v_1, \ldots, v_6\}$ rather than in terms of $\{b_1, \ldots, b_6\}$ and take the appropriate linear combinations of these equations, so that when we reduce them modulo two, the equations of the reduction define the equations of the Jacobian. This is done through the Smith normal form-like procedure that we previously explained. These computations can be found in the notebook `Part 2`, and the equations are in the text file `72 equations of the Jacobian.txt`. This embedding can also be accessed in Magma through the function `GeneralJacobianSurface` in `Functions.m`, as well as many other functions that connect this projective model with the machinery already implemented in Magma to work with Jacobians.

4.4.3 Computing equations of Kummer surfaces and their desingularisations in characteristic two

The k_i that we have defined generate an embedding of the Kummer surface X into $\mathbb{P}^3$ given by the vanishing of a quartic polynomial. When we reduce this polynomial modulo two, this matches the variety found by Duquesne [Duq10], which precisely have the right singularities described by Katsura [Kat78]: four D_4^1 singularities in the ordinary case, two D_8^2 singularities in the almost ordinary case, and one $④^1_{0,1}$ singularity in the supersingular case. As described in section 4.2, the rational map

$$\begin{aligned} \mathrm{Jac}(\mathcal{C}) &\dashrightarrow \mathbb{P}^5 \\ D &\longmapsto [b_1(D) : \cdots : b_6(D)] \end{aligned}$$

induces a closed embedding of a Kummer surface Y inside $\mathbb{P}^5$ as the complete intersection of three quadrics and there is a degree four birational map from X to Y which is defined outside of the singular locus of X. The scheme Y can be accessed in Magma via the function `DesingularisedKummer`.

Now, consider the reduction of the functions b_i in the reduced curve $\overline{\mathcal{C}}$, which we will denote by $\overline{b}_i$. While these are still linearly independent by construction, the first big difference with respect to the characteristic zero case is that, while in characteristic zero the b_i did not belong to $\mathrm{Sym}^2\mathcal{L}(\Theta_+ + \Theta_-)$, the space of quadratic functions in $\{k_1, k_2, k_3, k_4\}$, all the $\overline{b}_i$ can be expressed as quadratic functions in the $\{\overline{k}_1, \overline{k}_2, \overline{k}_3, \overline{k}_4\}$.

The rational map

$$\begin{aligned} \mathrm{Jac}(\mathcal{C}) &\dashrightarrow \mathbb{P}^5 \\ D &\longmapsto [\bar{b}_1(D) : \cdots : \bar{b}_6(D)] \end{aligned}$$

defines an embedding of a Kummer surface Y inside $\mathbb{P}^5$ as the complete intersection of three quadrics (A.1.5), but unlike in characteristic zero, this surface Y is not smooth. However, this map is still of interest, as all the $\bar{b}_i$ are simultaneously zero precisely at the points corresponding to $\mathrm{Jac}(\mathcal{C})[2]$, and thus the indeterminacy locus of the map

$$\begin{aligned} X &\dashrightarrow Y \\ [\bar{k}_1 : \cdots : \bar{k}_4] &\longmapsto [\bar{b}_1 : \cdots : \bar{b}_6] \end{aligned}$$

coincides with the singular locus of X. The inverse of this map, which we will denote by φ is a blow-up of the singular locus, which will be analysed in section 4.5.

We have computed explicit equations (A.1.6) defining this map, coming from the fact that the function $(2y_1 + g(x_1))(2y_2 + g(x_2))\bar{k}_i$ can be expressed as a polynomial in $\{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4\}$. The fact that this map involves only the first four $\bar{b}_i$ implies that the projection map from $\mathbb{P}^5$ to $\mathbb{P}^3$ consisting of taking the first four coordinates descends into a rational map

$$\begin{aligned} Y &\dashrightarrow W \subset \mathbb{P}^3 \\ [\bar{b}_1 : \cdots : \bar{b}_6] &\longmapsto [\bar{b}_1 : \cdots : \bar{b}_4] \end{aligned}$$

where W is a quartic surface in $\mathbb{P}^3$ which, by similarity with the characteristic zero case, we will call the Weddle surface (A.1.7). We will analyse its features according to the p-rank of the curve in the section 4.6.

4.5 Partial desingularisations of Kummer surfaces in characteristic two

In order to describe what the partial desingularisations look like, it will be convenient to analyse separately the cases according to the p-rank. The following proposition will be useful:

Proposition 4.5.1. *Let $\mathcal{C}$ be a genus two curve of the form $y^2 + g(x)y = f(x)$ with $\deg(g) = 3$. Then, $\mathrm{Jac}(\mathcal{C})(C)$ is ordinary, almost ordinary or supersingular, according to whether $g(x)$ has three, two or one distinct roots.*

Proof. As in the characteristic zero case, it is easy to see that any non-zero 2-torsion point is of the form $D_{ij} = (w_i) + (w_j) - K_{\mathcal{C}}$ where $\{w_i, w_j\}$ is an unordered pair of Weierstrass points of $\mathcal{C}$. Every non-trivial Weierstrass point of $\mathcal{C}$ is preserved by the hyperelliptic involution, and so, in characteristic two, as ι sends (x, y) to $(x, y + g(x))$, we deduce that (x, y) is a Weierstrass if and only if x is a root of $g(x)$.

Therefore, over the splitting field of g, there are $\binom{3}{2} = 3$ non-trivial 2-torsion points if and only if g has three distinct roots, $\binom{2}{2} = 1$ non-trivial 2-torsion points if and only if g has two distinct roots and no non-trivial 2-torsion points if g only has one root. □

For models of genus two curves with $\deg(g) < 3$, similar results can be found. However, given a genus two curve over a field of characteristic two described by the model $y^2 + g(x)y = f(x)$, we can find an isomorphism to a model of the same form with $\deg(g) = 3$ defined over the field of definition, by considering a morphism that maps the Weierstrass point of infinity to another point of the curve, and does not map any of the Weierstrass points to infinity[1]. We implemented this in Magma as the function `GenusTwoModel`.

We now describe the geometry. All equations for the curves and surfaces discussed here were computed in the Mathematica notebook `Part 3` and are available in Magma via the function `Lines`. These equations can be used to verify the accuracy of the following sections, as demonstrated in `Examples.m`.

4.5.1 The geometry of the ordinary case

Let $\mathcal{C}$ be an ordinary genus two curve of the form

$$y^2 + (\sum_{j=0}^{3} g_j x_i)y = \sum_{j=0}^{6} f_j x_i,$$

so that the Weierstrass points have coordinates (α_i, β_i), where $1 \le i \le 3$ and β_i is $\sqrt{\sum_{j=0}^{6} f_j \alpha^i}$. Note that, by Proposition 4.5.1, these α_i correspond to the three distinct roots of g. As in the characteristic zero case, the 2-torsion points of $\mathrm{Jac}(\mathcal{C})$ are of the form $D_{ij} = (w_i) + (w_j) - K_{\mathcal{C}}$ where $\{w_i, w_j\}$ are Weierstrass points whose coordinates are (α_i, β_i) and (α_j, β_j), and each of these corresponds to a singular point P_{ij} of the Kummer surface X associated to $\mathcal{C}$.

Similarly to the characteristic zero case, these singular points are defined over the field $k(\alpha_i + \alpha_j, \alpha_i\alpha_j)$. In fact, the equations of these points in our model are given by

$$P_O = [0:0:0:1], \qquad P_{ij} = \left[1 : \alpha_i + \alpha_j : \alpha_i\alpha_j : \frac{f_1 + \alpha_i\alpha_j f_3 + \alpha_i^2\alpha_j^2 f_5}{\alpha_i + \alpha_j}\right].$$

In characteristic two, it still makes sense to talk about tropes in X: we can define T_i to be the image of $\mathcal{C} \times \{w_i\}$ under the composition of the maps $\mathcal{C}^{(2)} \to \mathrm{Jac}(\mathcal{C})$ and $\mathrm{Jac}(\mathcal{C}) \to X$. Then, T_i is a conic in X, which goes through the points P_O, P_{ij} and P_{ik} where the indices $\{i, j, k\}$ are all distinct.

[1] There is actually an exception to this, which is the case when the field of definition is $\mathbb{F}_2$ and the curve is ordinary, as in this case there may not be enough elements in $\mathbb{F}_2$ to find this morphism over the field of definition.

Note that this trope could also be defined in an alternative way by considering the unique plane going through P_O, P_{ij} and P_{ik} (whenever the roots of g are distinct, these points are not collinear), which intersects X in the conic T_i with multiplicity two. This way, we can define a fourth trope, which we will denote by T_{123}, as the conic in X going through the singular points P_{12}, P_{13} and P_{23}.

In the same way as in the characteristic zero case, the action in the Jacobian induced by the translation by a 2-torsion point D_{ij} descends to a linear action τ_{ij} on the Kummer surface, which permutes the tropes according to the rules:

$$\tau_{ij}(T_i) = T_j, \qquad \tau_{ij}(T_{123}) = T_k,$$

where $\{i, j, k\}$ are all distinct indices. In our model, the tropes are defined by the intersection of X with the following planes:

$$\begin{aligned}
\pi_1 &= \alpha_1^2\overline{k}_1 + \alpha_1\overline{k}_2 + \overline{k}_3 = 0,\\
\pi_2 &= \alpha_2^2\overline{k}_1 + \alpha_2\overline{k}_2 + \overline{k}_3 = 0,\\
\pi_3 &= \alpha_3^2\overline{k}_1 + \alpha_3\overline{k}_2 + \overline{k}_3 = 0,\\
\pi_{123} &= (f_1 + f_3 + f_5)g_2^2\overline{k}_1 + g_2(f_5g_1 + f_1g_3 + f_3g_3)\overline{k}_2 + (f_5g_1^2 + f_3g_1g_3 + f_1g_3^2)\overline{k}_3 + g_2(g_1 + g_3)\overline{k}_4 = 0.
\end{aligned}$$

The number of tropes and singular points defined over the ground field also depends on how the polynomial $g(x)$ decomposes into irreducible factors.

Partition	# tropes of type T_i	# tropes of type T_{ijk}	# singular points
$\{1, 1, 1\}$	3	1	4
$\{1, 2\}$	1	1	2
$\{3\}$	0	1	1

Table 4.4: Number of tropes and singular points defined over the base field in characteristic two.

There is another possible description of these tropes from specialisation from the characteristic zero case. Consider a curve $\mathcal{C}$ defined over a discrete valuation ring whose fraction field K is complete and with a perfect residue field of characteristic two, such that all the 2-torsion is defined over K and such that $\mathcal{C}$ has good ordinary reduction.

It is easy to check that the Weierstrass points of $\mathcal{C}$ are a closed subvariety of $\mathcal{C}$ whose x-coordinates are the roots of the polynomial $4f(x) + g(x)^2$. From this, we can see that these reduce 2-to-1 to the Weierstrass points of $\overline{\mathcal{C}}$ whose x-coordinates are roots of $g(x)$. Moving on to the Jacobian, this phenomenon shows as a reduction 4-to-1 of the 2-torsion points, as closed points have to reduce to closed points.

In the Kummer, this shows as well as a reduction 4-to-1 of the singular points, which can be seen from the fact that, in the explicit models of the Kummer that we have computed, the singular locus of X reduces to the singular locus of the Kummer surface of the reduced curve.

But also, it manifests in the surface as a reduction 4-to-1 of the tropes in a natural way: the reduction of each trope is the corresponding trope that goes through all the reductions of the singular points.

Now consider the blow-up φ that was described in the previous section. The exceptional divisors associated to the resolution of a D_4^1 singularity form a tree configuration. For each of the four D_4^1 singularities, the partial desingularisation map blows up the central exceptional curve of each of the four D_4^1 singularities; therefore, the partial desingularisation has twelve A_1 singularities.

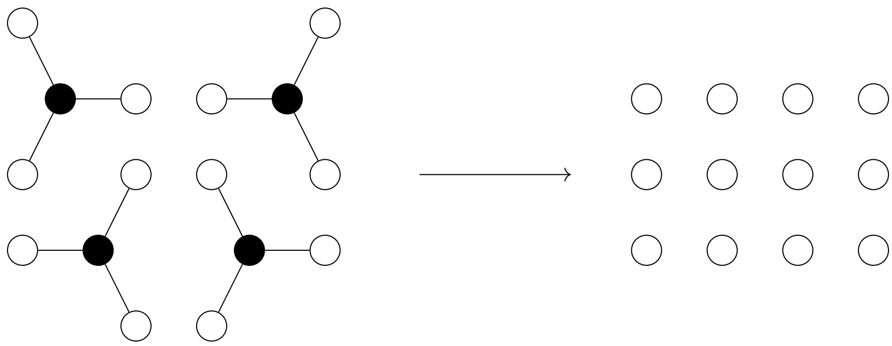

Figure 4.1: Partial desingularisation of the Kummer surface in the ordinary case.

From the explicit equations that we have computed, it is easy to check that the image of each of the conics corresponding to the tropes of X is a line of Y.

Then, these twelve singularities are nodes that lie in the intersection points of the four exceptional divisors associated with the singularities of the Kummer surface, and the image of the four tropes.

We can observe that all tropes and exceptional lines of Y lie in the hyperplane section of Y:

$$(\alpha_2^2\alpha_3\beta_1 + \alpha_2\alpha_3^2\beta_1 + \alpha_1^2\alpha_3\beta_2 + \alpha_1\alpha_3^2\beta_2 + \alpha_1^2\alpha_2\beta_3 + \alpha_1\alpha_2^2\beta_3)\bar{b}_1 + (\alpha_2^2\beta_1 + \alpha_3^2\beta_1 + \alpha_1^2\beta_2 + \alpha_3^2\beta_2 + \alpha_1^2\beta_3 + \alpha_2^2\beta_3)\bar{b}_2$$
$$+(\alpha_2\beta_1 + \alpha_3\beta_1 + \alpha_1\beta_2 + \alpha_3\beta_2 + \alpha_1\beta_3 + \alpha_2\beta_3)\bar{b}_3 + (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3)\bar{b}_4 = 0.$$

As described by Katsura and Kondō, if we denote by $\{E_O, E_{12}, E_{13}, E_{23}\}$ the exceptional divisors corresponding to the singular points of X and by $\{\hat{T}_1, \hat{T}_2, \hat{T}_3, \hat{T}_{123}\}$ the images of the tropes in Y, then these divisors intersect according to the following configuration:

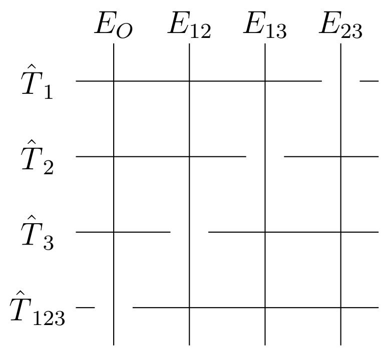

Figure 4.2: Intersections of the tropes and exceptional divisors in the ordinary case.

From this reasoning, we can deduce that the minimal resolution of the Kummer surface contains twenty (-2)-curves, which are the proper transforms of the eight lines described above and the twelve exceptional curves that we obtain from blowing-up the singular points. These curves meet according to the following dual graph:

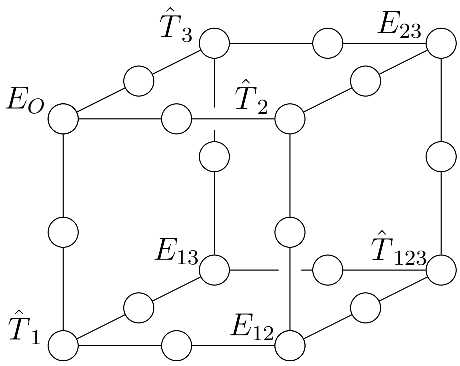

Figure 4.3: Dual graph of the resolution of the singular points and the tropes in the ordinary case.

Katsura and Kondō showed that a general Kummer surface in $\mathbb{P}^3_{x,y,z,t}$ can be described by the equation

$$\begin{aligned}(a_1+a_2)^2(c_3x^2y^2+d_3z^2t^2)+(a_1+a_3)^2(c_2x^2z^2+d_2y^2t^2)\\ +(a_2+a_3)^2(c_1x^2t^2+d_1y^2z^2)+(a_1+a_2)(a_2+a_3)(a_3+a_1)xyzt=0.\end{aligned} \tag{4.5.1}$$

In this model, the planes defining the tropes of the Kummer are given by the equations $x=0$, $y=0$, $z=0$ and $t=0$.

The linear projective map ψ defined by

$$\psi([\overline{k}_1 : \overline{k}_2 : \overline{k}_3 : \overline{k}_4]) = [\pi_1 : \pi_2 : \pi_3 : \pi_{123}]$$

is an isomorphism between X and the variety defined in equation (4.5.1) with the parameters given in A.1.8. Katsura and Kondō defined a Cremona transformation ϕ in their model of the Kummer, by setting

$$\phi([x : y : z : t]) = \left[\sqrt{d_1 d_2 d_3}\, yzt : \sqrt{c_1 c_2 d_3}\, xzt : \sqrt{c_1 d_2 c_3}\, xyt : \sqrt{d_1 c_2 c_3}\, xyz\right]$$

and this induces a Cremona transformation in our model by considering the composition of maps $\psi^{-1} \circ \phi \circ \psi$. Similarly, they described the linear actions τ_{ij} induced by the addition by a 2-torsion on $\mathrm{Jac}(\mathcal{C})$, and we can use these to find the equations for our model.

They also described the partial desingularisation of the equation (4.5.1), as a complete intersection described by the equations:

$$\sum_{i=1}^{3} X_i Y_i = \sum_{i=1}^{3} a_i X_i Y_i + c_i X_i^2 + d_i Y_i^2 = \sum_{i=1}^{3} a_i^2 X_i Y_i = 0.$$

We can also connect this model of partial desingularisation to Y through the change of variables given in A.1.9. Once again, Katsura and Kondō described three automorphisms ι_1, ι_2, ι_3 in the model they developed corresponding to the generators of the group $(\mathbb{Z}/2\mathbb{Z})^3$ and through the change of coordinates, these correspond to the linear actions in Y corresponding to the translation by a 2-torsion point, and the Cremona transformation which interchanges tropes with exceptional divisors.

4.5.2 The geometry of the almost ordinary case

In this case, by Proposition 4.5.1, $g(x)$ has two distinct roots over the splitting field of g, one with multiplicity one which we will denote by α_1, and one with multiplicity two, which we will denote by α_2. It may not be obvious from the start how the asymmetry between these two roots affects the geometry of the surfaces, but it will become apparent later.

In this case, the only non-trivial 2-torsion point of $\mathrm{Jac}(\mathcal{C})$ corresponds to the divisor $D_{12} = (w_1) + (w_2) - K_{\mathcal{C}}$ where w_1 and w_2 are the Weierstrass points (α_1, β_1) and (α_2, β_2). This point corresponds to a singular point P_{12} of the Kummer surface X associated to $\mathcal{C}$, which, in addition to the point associated to the identity in the group law P_O, are the two D_8^2 singularities of X. Assuming that we are working over a perfect field, these points are always defined over k, the ground field of $\mathcal{C}$, as

$$g(x) = g_3(x - \alpha_1)(x - \alpha_2)^2 = g_3 x^3 + g_3 \alpha_1 x^2 + g_3 \alpha_2^2 x + g_3 \alpha_1 \alpha_2^2.$$

Therefore,

$$\alpha_1 = \frac{g_2}{g_3}, \qquad\qquad \alpha_2 = \sqrt{\frac{g_1}{g_3}}.$$

As before, we can define T_i to be the image of $\mathcal{C} \times \{w_i\}$ under the composition of the maps $\mathcal{C}^{(2)} \to \mathrm{Jac}(\mathcal{C})$ and $\mathrm{Jac}(\mathcal{C}) \to X$. Then, T_1 and T_2 are conics in X defined over k, which go through the points P_O and P_{12}. An easy way of computing the equations for these is from specialisation from the ordinary case by considering a general equation of an ordinary curve of the form

$$y^2 + g_3(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = f(x),$$

and setting α_3 to be equal to α_2. Then,

- P_O and P_{23} both specialise to P_O.
- P_{12} and P_{13} both specialise to P_{12}.
- T_1 and T_{123} both specialise to T_1.
- T_2 and T_3 both specialise to T_2.

Through this description, we see that T_1 and T_2 meet P_O and P_{12} with different multiplicity as, for instance in the case of T_1, what happens is that T_1 and T_{123} both go through P_{12} and P_{13}, which reduce to P_{12}, and through another point which reduces to P_O. Therefore, T_1 goes through P_{12} with a greater multiplicity than P_O. Through a similar reasoning we can see that T_2 goes through P_O with a greater multiplicity than P_{12} and this plays a role on the singularities that we obtain when we blow up X.

We can see that there is a specialisation 2-to-1 with respect of the ordinary case, or, if instead we wanted to study this almost ordinary case as a reduction from characteristic zero, it would be a reduction 8-to-1 of both tropes and singular points. Now, consider the blow-up that was described in 4.4.3. The exceptional divisors associated with the resolution of a D_8^2 singularity form a tree configuration and the partial desingularisation map blows up one of the central exceptional curves of each of the two D_8^2 singularities. As a consequence, each D_8^2 gets blown-up into a D_4^0 and an A_3 singularity.

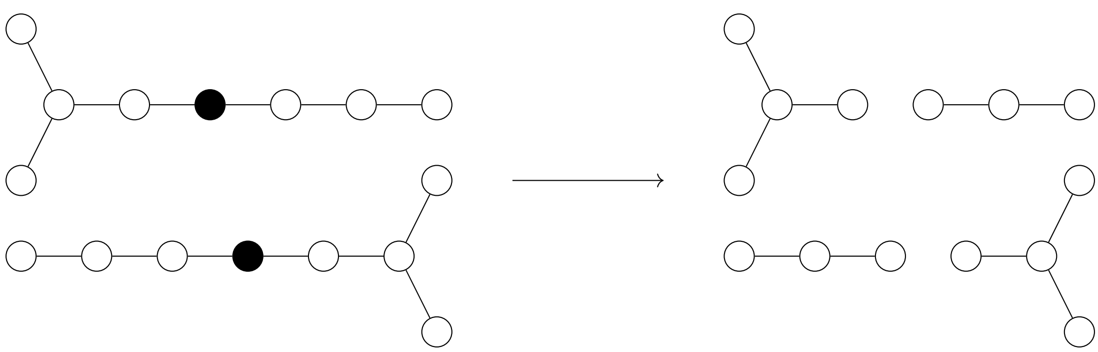

Figure 4.4: Partial desingularisation of the Kummer surface in the almost ordinary case.

If we denote by $\{E_O, E_{12}\}$ the exceptional divisors corresponding to the singular points of X and by $\{\hat{T}_1, \hat{T}_2\}$ the tropes, then, E_O and E_{12} intersect both $\hat{T}_1$ and $\hat{T}_2$, and the four points of intersection correspond to the four singular points of Y, where the D_4^0 singularities correspond to the intersection of the tropes with the singular points with the greatest multiplicities in X ($E_O \cap \hat{T}_2$ and $E_{12} \cap \hat{T}_1$) and the A_3 singularities correspond to the other two ($E_O \cap \hat{T}_1$ and $E_{12} \cap \hat{T}_2$).

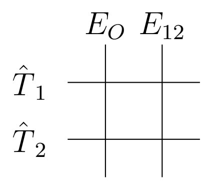

Figure 4.5: Intersections of the tropes and exceptional divisors in the almost ordinary case.

Furthermore, we can deduce that the minimal resolution of the Kummer surface contains eighteen (-2)-curves which are the proper transforms of the four lines described above, and fourteen coming from the desingularisation of the A_3 and D_4^0 singularities. The intersection graph of these curves is given by the following diagram:

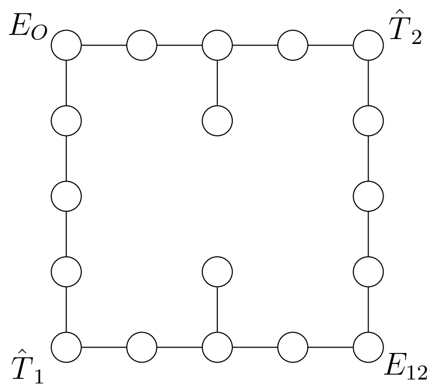

Figure 4.6: Dual graph of the resolution of the singular points and the tropes in the almost ordinary case.

A justification for why the curves intersect in this way will be provided in the next section.

As in the ordinary case, Katsura and Kondō proved that every Kummer surface associated to an almost ordinary abelian surface admits a model as a quartic in $\mathbb{P}^3_{x,y,z,t}$ of the form:

$$\begin{aligned}
&b_3^2c_1x^4 + b_2^2d_1y^4 + b_1^2d_1z^4 + b_4^2c_1t^4\\
&+ \left(b_3^2d_2 + b_2^2c_2 + (a_1+a_2)^2c_3 + (a_1+a_2)\,b_2b_3\right)x^2y^2\\
&+ \left(b_3^2d_3 + b_1^2c_3 + (a_1+a_2)^2c_2 + (a_1+a_2)\,b_1b_3\right)x^2z^2\\
&+ \left(b_2^2d_3 + b_4^2c_3 + (a_1+a_2)^2d_2 + (a_1+a_2)\,b_2b_4\right)y^2t^2\\
&+ \left(b_1^2d_2 + b_4^2c_2 + (a_1+a_2)^2d_3 + (a_1+a_2)\,b_1b_4\right)z^2t^2\\
&+ (a_1+a_2)^2\left(b_3x^2yz + b_2xy^2t + b_1xz^2t + b_4yzt^2\right) = 0.
\end{aligned}$$

One can also relate our model to theirs through a change of coordinates, as in the ordinary case. This change of coordinates is quite lengthy and it is described in the notebook `Part 3`. We did not need this to describe the automorphisms in our model for the Kummer surface, as we can specialise from the ordinary model into the almost ordinary model simply by setting $\alpha_3 = \alpha_2$ and $\beta_3 = \beta_2$.

Through a change of coordinates, one can use this to find the equations for the automorphisms in Katsura and Kondō's model of an almost ordinary quartic Kummer surface, for instance, the Cremona transformation which they were not able to compute. This transformation can be described by the transformation

$$\phi([x:y:z:t]) = [x':y':z':t']$$

where

$$\begin{aligned}
x' &= \sqrt{d_1}x\left(\sqrt{b_2}y + \sqrt{b_1}z\right)^2,\\
y' &= \sqrt{c_1}y\left(\sqrt{b_3}x + \sqrt{b_4}t\right)^2,\\
z' &= \sqrt{c_1}z\left(\sqrt{b_3}x + \sqrt{b_4}t\right)^2,\\
t' &= \sqrt{d_1}t\left(\sqrt{b_2}y + \sqrt{b_1}z\right)^2.
\end{aligned}$$

Specialising the partially desingularised model that we computed of Y, we can also connect this with the model in $\mathbb{P}^5$ of Katsura and Kondō.

4.5.3 The geometry of the supersingular case

In the supersingular case, $g(x)$ has only one root over the splitting field of g, which we will denote by α_1, and it does not have any non-trivial 2-torsion points. The point in the Kummer surface corresponding to the identity in the abelian surface is an elliptic singular point of type $④^1_{0,1}$ which in our model X corresponds to the coordinates $[0:0:0:1]$.

Even though there are no 2-torsion points, there is still a Weierstrass point w_1, corresponding to the point (α_1, β_1) and, as before, we can define a trope T_1 to be the image of $\mathcal{C} \times \{w_1\}$ under the composition of the maps $\mathcal{C}^{(2)} \to \mathrm{Jac}(\mathcal{C})$ and $\mathrm{Jac}(\mathcal{C}) \to X$. This can be found to be a specialisation 2-to-1 with respect to the almost ordinary case by making α_2 tend to α_1, or, alternatively, as a reduction 16-to-1 of the tropes and singular points.

Now consider the blow-up that was described in section 4.4.3. In the supersingular case, the singular point $④^1_{0,1}$ is a contraction of five lines in a tree configuration in which the central (-2)-curve has multiplicity two and the other four curves are three (-2)-curves and one (-3)-curve. Then, the desingularisation map corresponds to the following transformation [Sch09, Theorem 6.3]:

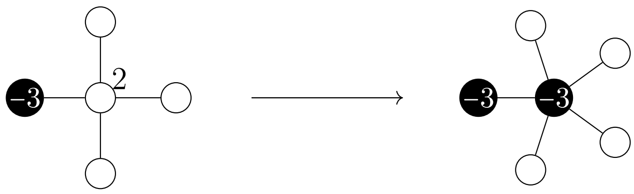

Figure 4.7: Blow-up relating the singularities $④^1_{0,1}$ and $A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o}$.

If we denote by E_O the exceptional divisor corresponding to the singular point of X and by $\hat{T}_1$ the trope, then the singularity lies precisely in the intersection of both lines. As in the previous cases, there is a Cremona transformation in Y exchanging E_O and $\hat{T}_1$, and both this and the corresponding transformation in X can be easily described in our model.

In the supersingular case, the desingularisation scheme model that was found by Katsura and Kondō has the following form:

$$
\begin{gathered}
\left(b_3^2c_1 + b_7^2c_3\right)x^4 + \left(b_2^2d_1 + b_8^2c_3\right)y^4 + \left(b_1^2d_1 + b_6^2d_3\right)z^4 + \left(b_4^2c_1 + b_5^2d_3\right)t^4 \\
+b_5\left(b_1b_5 + b_4b_7\right)xt^3 + b_7\left(b_2b_7 + b_3b_5\right)x^3t + b_2\left(b_2b_6 + b_3b_8\right)xy^3 + b_8\left(b_2b_6 + b_3b_8\right)y^3z \\
+b_3\left(b_2b_7 + b_3b_5\right)x^3y + b_4\left(b_1b_5 + b_4b_7\right)zt^3 + b_6\left(b_1b_8 + b_4b_6\right)yz^3 + b_1\left(b_1b_8 + b_4b_6\right)z^3t \\
+\left(b_2^2c_2 + b_3^2d_2\right)x^2y^2 + \left(b_1^2c_3 + b_3^2d_3 + b_6^2c_1 + b_7^2d_1\right)x^2z^2 + \left(b_5^2c_2 + b_7^2d_2\right)x^2t^2 \\
+\left(b_6^2d_2 + b_8^2c_2\right)y^2z^2 + \left(b_2^2d_3 + b_4^2c_3 + b_5^2d_1 + b_8^2c_1\right)y^2t^2 + \left(b_1^2d_2 + b_4^2c_2\right)z^2t^2 \\
+b_7\left(b_2b_6 + b_3b_8\right)x^2yz + b_3\left(b_1b_5 + b_4b_7\right)x^2zt + b_8\left(b_2b_7 + b_3b_5\right)xy^2t + b_2\left(b_1b_8 + b_4b_6\right)y^2zt \\
+b_1\left(b_2b_6 + b_3b_8\right)xyz^2 + b_6\left(b_1b_5 + b_4b_7\right)xz^2t + b_4\left(b_2b_7 + b_3b_5\right)xyt^2 + b_5\left(b_1b_8 + b_4b_6\right)yzt^2 = 0.
\end{gathered}
$$

Our model is slightly simpler, as it can be described by specialising from the almost ordinary case by substituting in the equation α_2 by α_1 and β_2 by β_1. For the other two cases, it was relatively easy to relate our model to Katsura and Kondō's, as sending the tropes to the tropes and the singular points to the singular points provided enough information to almost match both sets of equations.

However, for the supersingular case, as there are only one singular point and one trope, we could not find a change of variables which matched our model with Katsura and Kondō's.

4.6 Weddle surfaces and blow-ups of the exceptional lines

Since they were first studied, one of the key features of quartic Kummer surfaces was the fact that, over algebraically closed fields, they were isomorphic to their projective dual. As a result, projecting away from a singular point gives rise to birationally equivalent quartic surfaces known as Weddle surfaces.

In characteristic zero, the construction of these surfaces is the following. As described in section 4.4, given a model of a Kummer surface in $\mathbb{P}^3$ as a quartic surface with sixteen nodes, we can construct a blow-up as the intersection in $\mathbb{P}^5$ of three quadrics. As this blow-up is a birational map, we can construct an inverse map, which is well-defined outside of the singular locus of X. Furthermore, as this map only depends on the four first coordinates b_1, b_2, b_3, b_4, the projection map of the first four coordinates $\mathbb{P}^5 \dashrightarrow \mathbb{P}^3$, defines a map from Y into $\mathbb{P}^3$, such that the closure of its image is given by a quartic surface $W \subset \mathbb{P}^3$ known as the **Weddle surface**.

After noticing that this map is well-defined outside of $b_1 = b_2 = b_3 = b_4 = 0$, which is precisely the exceptional line E_O associated to the identity in the Jacobian, one can check that the Weddle surface geometrically corresponds to the map π_O that consists of projecting Y away from E_O.

In any characteristic different from two, this transformation contracts the tropes $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$, $\hat{T}_4$, $\hat{T}_5$ and $\hat{T}_6$, which are the ones meeting E_O, into singular points of W of type A_1, which we will denote by Q_i. The expression in coordinates for these points in our model are given by

$$Q_i = \left[1 : \omega_i : \omega_i^2 : \frac{g(\omega_i)}{2}\right].$$

From this description we deduce that, as the coordinates of the Q_i depend exclusively of the Weierstrass points, one can use these points to recover our initial curve $\mathcal{C}$ from the equation of the Weddle.

The images of the other exceptional lines E_{ij} and tropes $\hat{T}_{ijk}$ are also lines in the Weddle surface, and they have a very nice geometric description [Moo28].

All the singular points Q_i are in general position meaning that no four of them lie in the same plane[2], and if we consider the plane going through three of these singular points, say Q_i, Q_j and Q_k, then the intersection of this plane with the Weddle surface always consists of the union of $\pi_O(E_{ij})$, $\pi_O(E_{ij})$, $\pi_O(E_{ij})$, and $\pi_O(\hat{T}_{ijk})$.

Furthermore, there is a very special rational curve which we will denote by C going through all the singular points, which is a twisted cubic defined by the equations:

$$b_2^2 - b_1b_3 = -2b_2b_4 + b_1b_2g_0 + b_2^2g_1 + b_2b_3g_2 + b_3^2g_3 = -2b_1b_4 + b_1^2g_0 + b_1b_2g_1 + b_1b_3g_2 + b_2b_3g_3 = 0.$$

We now consider the blow-up of the line E_O in Y, which is closely related to the Weddle surface. As Y is smooth and E_O is a smooth subvariety of it, the blow-up is isomorphic to Y, so no new information is gained from this in characteristic zero. However, understanding the blow-up process will help us understand the blow-up of the exceptional lines in the specialisation in characteristic two.

The blow-up scheme of E_O, $\mathrm{Bl}_{E_O}(Y)$ is the Zariski closure of the image of the graph morphism

$$\begin{aligned} \Gamma_{\pi_O}: \quad & Y \dashrightarrow Y \times \mathbb{P}^3 \\ [b_1 : b_2 : b_3 : b_4 : b_5 : b_6] & \longmapsto [b_1 : b_2 : b_3 : b_4 : b_5 : b_6] \times [b_1 : b_2 : b_3 : b_4]. \end{aligned}$$

Let $\varphi_O : \mathrm{Bl}_{E_O}(Y) \to Y$ be the blow-up map. We can easily see that $\mathrm{Bl}_{E_O}(Y) \subseteq Y \times W$, and we can therefore describe the subvarieties of $\mathrm{Bl}_{E_O}(Y)$ as the restriction to $\mathrm{Bl}_{E_O}(Y)$ of subvarieties of $Y \times W$. Then, we can see what happens to the pullbacks of all the exceptional lines and tropes of Y under φ_O:

$$\begin{aligned} \varphi_O^*(E_O) &= E_O \times C, & \varphi_O^*(E_{ij}) &= E_{ij} \times \pi_O(E_{ij}), \\ \varphi_O^*(\hat{T}_i) &= \hat{T}_i \times Q_i, & \varphi_O^*(\hat{T}_{ijk}) &= \hat{T}_{ijk} \times \pi_O(\hat{T}_{ijk}). \end{aligned}$$

While the map φ_O that we just described is special, in the sense that it is always defined over the field of definition of the curve and does not depend on the curve, it is important to bear in mind that projecting away from any of the 32 lines of Y (the sixteen tropes or the sixteen exceptional lines) would also give us a map from Y into a quartic surface in $\mathbb{P}^3$ with the same singularities. Any of these maps can be described as $\tau \circ \pi_O$, where τ is any automorphism of Y exchanging the trope that we are projecting and E_O.

We will now see what happens when the field of definition has characteristic two, and describe the resulting singularities of the Weddle surface and what we obtain when we blow up the exceptional lines.

[2]This can easily be seen from the description in coordinates of the singular points, as the matrix of the coordinates of any four points can be changed by a linear change of coordinates to a Vandermonde matrix, and therefore its determinant is never zero as all the ω_i are different.

It is worth mentioning that, in a recent article, Dolgachev [Dol23] generalised the notion of Weddle surface for fields of characteristic two by defining them as the locus of singular points in the web of quadrics going through a set of six points of the form Q_i. The notion of Weddle surface we will refer to in the next subsections is different and corresponds to the specialisation of a Weddle surface in characteristic zero to characteristic two, that is, the surface obtained when we project away from the exceptional line E_O in Y.

4.6.1 The ordinary case

The Weddle surface associated to an ordinary genus two curve has three A_3 singularities and four A_1 singularities. Projecting away from E_O does two things. Firstly, it blows up the singular points that are in the intersection of the tropes $\hat{T}_1, \hat{T}_2, \hat{T}_3$ with E_O into three lines L_1, L_2 and L_3 and, secondly, it contracts these tropes.

As each of the tropes contains two singular points, these tropes are contracted to A_3 singularities. If we denote these singularities by Q_i, it is easy to check that the coordinates of these Q_i in our model are given by

$$Q_i = [1 : \alpha_i : \alpha_i^2 : \beta_i].$$

In addition to these three singular points, there are also four additional singular points of type A_1. One of them, which we will denote Q_O, has coordinates $Q_O = [0 : 0 : 0 : 1]$ and corresponds to the contraction of E_O under π_O. The other three correspond to the images under the projection map π_O of the three singularities of Y that lie in $\hat{T}_{123}$, that is, they are in $\pi_O(E_{ij} \cap \hat{T}_{123})$. We will denote these by Q_{ij}.

Similarly to the characteristic zero case, we can recover both the coordinates of the Weierstrass points and the curve we started with, from the singular points.

Another curious fact is that all singular points of the Weddle surface except for Q_O lie in the same plane, which is given by the equation

$$(\alpha_2\alpha_3(\alpha_2+\alpha_3)\beta_1 + \alpha_1\alpha_3(\alpha_1+\alpha_3)\beta_2 + \alpha_2\alpha_3(\alpha_2+\alpha_3)\beta_1)\bar{b}_1 + ((\alpha_2+\alpha_3)^2\beta_1 + (\alpha_1+\alpha_3)^2\beta_2 + (\alpha_1+\alpha_2)^2\beta_3)\bar{b}_2$$
$$+((\alpha_2+\alpha_3)\beta_1 + (\alpha_1+\alpha_3)\beta_2 + (\alpha_1+\alpha_2)\beta_3)\bar{b}_3 + (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\bar{b}_4 = 0.$$

The intersection of this plane with the surface is the union of four lines corresponding to $\pi_O(\hat{T}_{123})$, $\pi_O(E_{12})$, $\pi_O(E_{13})$ and $\pi_O(E_{23})$ and is represented in the following diagram:

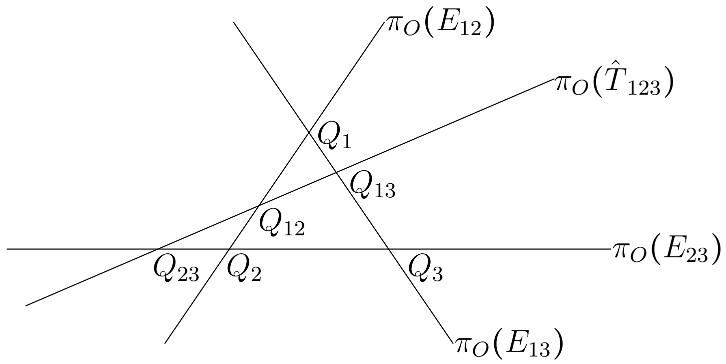

Figure 4.8: Intersection of lines inside the Weddle surface in the ordinary case.

Now, consider the blow-up of the curve E_O, $\varphi_O : \mathrm{Bl}_{E_O}(Y) \to Y$, which is defined in the exact same way as for fields of characteristic different from two. Then, the pullbacks of the exceptional lines and tropes of Y are given by

$$\varphi_O^*(E_O) = E_O \times Q_O, \qquad \varphi_O^*(E_{ij}) = E_{ij} \times \pi_O(E_{ij}),$$
$$\varphi_O^*(\hat{T}_i) = \hat{T}_i \times Q_i, \qquad \varphi_O^*(\hat{T}_{123}) = \hat{T}_{123} \times \pi_O(\hat{T}_{123}).$$

Furthermore, as we are blowing up E_O, which contained three singular points of type A_1 corresponding to the intersection of E_O with $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$, these singularities are resolved, and we have that if we let L_i to be the line in W going through Q_O and Q_i,

$$\varphi_O^{-1}(E_O \cap \hat{T}_i) = (E_O \cap \hat{T}_i) \times L_i.$$

None of the other singular points are resolved and we have that their preimages under the blow-up are

$$\varphi_O^{-1}(E_{ij} \cap \hat{T}_i) = (E_{ij} \cap \hat{T}_i) \times Q_i,$$
$$\varphi_O^{-1}(E_{ij} \cap \hat{T}_{123}) = (E_{ij} \cap \hat{T}_{123}) \times Q_{ij}.$$

Therefore, $\mathrm{Bl}_{E_O}(Y)$ has nine A_1 singularities. One can also understand $\mathrm{Bl}_{E_O}(Y)$ as a blow-up of the Weddle surface that resolves Q_O and blows up the central exceptional curve of each of the A_3 singularities that we denoted Q_i.

This is a diagram illustrating the blow-up process:

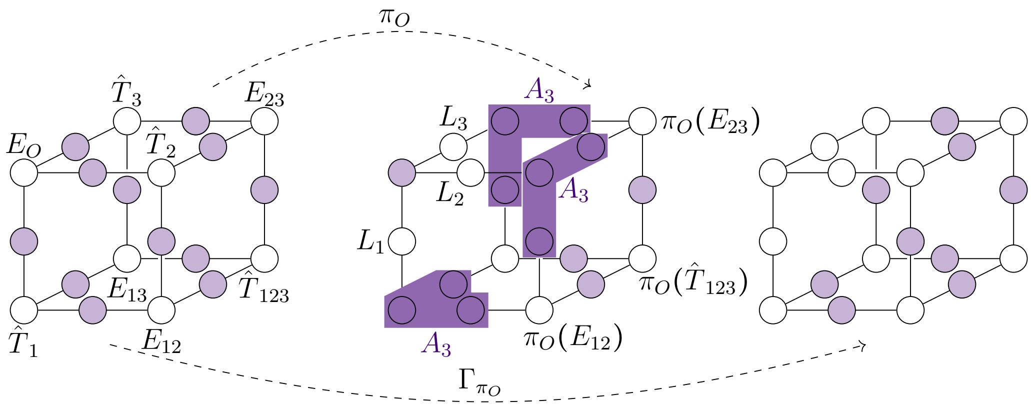

Figure 4.9: Blow-ups of the Kummer surface in the ordinary case.

A similar reasoning would apply if we blew up any of the other exceptional lines E_{ij}, which would result in resolving the three singular points contained in the line. The fact that this is the case can be used to construct an explicit model for the resolution of Y.

Proposition 4.6.1. *Let $\mathcal{I}_{ij} = \langle \mu_1^{(ij)}, \mu_2^{(ij)}, \mu_3^{(ij)}, \mu_4^{(ij)} \rangle$ be the ideal generated by four linear polynomials on the variables $\{\bar{b}_1, \ldots, \bar{b}_6\}$ such that $E_{ij} = \mathbb{V}(\mathcal{I}_{ij})$ in Y. Let π_{ij} be the morphism*

$$\begin{aligned} \pi_{ij} : \quad Y &\dashrightarrow \mathbb{P}^3 \\ [\bar{b}_1 : \bar{b}_2 : \bar{b}_3 : \bar{b}_4 : \bar{b}_5 : \bar{b}_6] &\longmapsto [\mu_1^{(ij)} : \mu_2^{(ij)} : \mu_3^{(ij)} : \mu_4^{(ij)}]. \end{aligned}$$

Then, the Zariski closure of the image of the graph morphism $\Gamma_{\pi_{ij}}$ corresponds to the blow-up scheme $\mathrm{Bl}_{E_{ij}}(Y)$ along the subvariety E_{ij}, which blows up the three singular points in E_{ij}. Furthermore, let $Z = E_O \cup E_{12} \cup E_{13} \cup E_{23}$ and consider the birational map $\phi : Y \dashrightarrow (\mathbb{P}^3)^4$, which acts in each copy of $\mathbb{P}^3$ as $\pi_O, \pi_{12}, \pi_{13}$ and π_{23} respectively. Then, the Zariski closure of the image of the graph morphism Γ_ϕ is the blow-up scheme $\mathrm{Bl}_Z(Y)$ and this is a resolution of the twelve A_1 singularities of Y.

Proof. As described in Subsection 4.5.1, the group $(\mathbb{Z}/2\mathbb{Z})^3$ acts linearly on Y. In particular, for every E_{ij}, there is a linear action τ_{ij} on Y of order two interchanging E_O and E_{ij}. As the image with respect of τ_{ij} of the ideal $\langle \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4 \rangle$ is $\mathcal{I}_{ij}$, we deduce that τ_{ij} induces a linear isomorphism between $\mathrm{Bl}_O(\tau_{ij}(Y))$ and $\mathrm{Bl}_{E_{ij}}(Y)$, showing that the construction of $\Gamma_{\pi_{ij}}$ corresponds to a blow-up of the exceptional line E_{ij}.

As for the second half of the statement, ϕ is a birational map that has a well-defined inverse away from Z. As the four exceptional lines are disjoint, in the open $Y \smallsetminus (Z \smallsetminus E_{ij})$, we have that the following diagram is commutative

$$\begin{tikzcd} & Y \smallsetminus (Z \smallsetminus E_{ij}) \arrow[dl, "\phi"'] \arrow[dr, "\pi_{ij}"] & \\ (\mathbb{P}^3)^4 \arrow[rr, "pr"] & & \mathbb{P}^3 \end{tikzcd}$$

and the projection map pr is an isomorphism between $\mathrm{im}(\phi)$ and $\mathrm{im}(\pi_{ij})$. As a consequence, we deduce that $\Gamma_\phi(Y \smallsetminus (Z \smallsetminus E_{ij})) \cong \Gamma_{\pi_{ij}}(Y \smallsetminus (Z \smallsetminus E_{ij}))$ and that in an open set not containing $Z \smallsetminus E_{ij}$, Γ_ϕ^{-1} is a blow-up of each of the E_{ij}. As these lines are all disjoint, we deduce that Γ_ϕ^{-1} blows up the union of all of the lines, and as all twelve singularities of Y lie in Z, and we have seen that the blow-up of each line resolves three of them, we deduce that $\mathrm{Bl}_Z(Y)$ resolves the twelve A_1 singularities. □

We can draw connections between the geometry in characteristic zero and two. Suppose Y is defined over a discrete valuation ring with a complete fraction field K and a perfect residue field of characteristic two k, such that all the 2-torsion is defined over K and such that $\mathcal{C}$ has good ordinary reduction.

Without any loss of generality, we assume that the roots $\{\omega_1, \omega_4\}$ of $f(x) + \frac{1}{4}g(x)^2$ reduce to α_1, $\{\omega_2, \omega_5\}$ reduce to α_2 and $\{\omega_3, \omega_6\}$ reduce to α_3. Letting $\overline{Y}$ and $\overline{E}_O$ denote the reduction of Y and E_O over the residue field, we can work out from our explicit model that the reduction of the scheme $\mathrm{Bl}_{E_O}(Y)$ gives us the scheme defining $\mathrm{Bl}_{\overline{E}_O}(\overline{Y})$.

Now, as we have explained before, the 2-torsion points of an ordinary abelian surface reduce 4-to-1 modulo two, so there are three other exceptional lines E_{14}, E_{25} and E_{36} in Y that reduce to $\overline{E}_O$. Moreover, in the Weddle surface W associated to Y,

- The twisted cubic E_O specialises to Q_O.
- The six singular points Q_i specialise to the three D_4^1 singularities $Q_{i \pmod 3}$ of $\overline{W}$.
- If i and j are not the same modulo 3, $\pi_O(E_{ij})$ specialises to $\pi_O(\overline{E}_{ij \pmod 3})$ and, otherwise, $\pi_O(E_{i(i+3)})$ specialises to L_i.
- If i, j and k are all different modulo 3, $\pi_O(\hat{T}_{ijk})$ specialises to $\pi_O(\hat{T}_{123})$.
- Otherwise, $\{\pi_O(\hat{T}_{125}), \pi_O(\hat{T}_{136})\}$ specialise to Q_1, $\{\pi_O(\hat{T}_{124}), \pi_O(\hat{T}_{145})\}$ specialise to Q_2 and $\{\pi_O(\hat{T}_{134}), \pi_O(\hat{T}_{146})\}$ specialise to Q_3.

Combining this description of how the Weddle surface specialises in the residue field with the previous description of the blow-up of E_O in characteristic two, we deduce that in $\mathrm{Bl}_{E_O}(Y)$, the pull-backs $\varphi_O^*(E_{14}), \varphi_O^*(E_{25})$ and $\varphi_O^*(E_{36})$ specialise to the exceptional lines corresponding to the singular points which get blown-up in $\mathrm{Bl}_{\overline{E}_O}(\overline{Y})$.

Therefore, in the Picard group of the desingularisation of the Kummer surface over K, inside the lattice $\oplus_{i=1}^{16} A_1$ formed by the sixteen exceptional lines, there is a sublattice formed by four lines $\oplus_{i=1}^{4} A_1$ that over the residue field k specialises to the sublattice D_4.

Furthermore, the previous description suggests that there is a configuration of four exceptional lines in Y reducing to $\overline{E}_O$, $\overline{E}_{12}$, $\overline{E}_{13}$ and $\overline{E}_{23}$, such that the blow-up of the union of these lines in Y specialises to the smooth model described in Proposition 4.6.1.

There are important constraints on what this configuration of lines has to be. As there is an action of $(\mathbb{Z}/2\mathbb{Z})^4$ in Y that must specialise to an action of $(\mathbb{Z}/2\mathbb{Z})^2$ in $\overline{Y}$, this forces our configuration of lines to be the orbit of E_O under a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Equivalently, the corresponding 2-torsion points must form a subgroup of $\mathrm{Jac}(\mathcal{C})[2](\overline{K})$. But also, if the exceptional lines are not all defined over K, which happens if the polynomial $f(x) + \frac{1}{4}g(x)^2$ does not fully split over K, the action of $\mathrm{Gal}(\overline{K}/K)$ on Y has to somehow be compatible with the action of $\mathrm{Gal}(\overline{K}/K)$ on $\overline{Y}$.

These observations suggest that studying the reduction of a Kummer surface at two over K relies on analysing the action of $\mathrm{Gal}(\overline{K}/K)$ on the 2-torsion of its associate abelian surface, and indeed we will see that this is the case in section 4.7.

4.6.2 The almost ordinary case

Moving on to the Kummer surface associated to an almost ordinary abelian surface over a field of characteristic two, we can check that the associated Weddle surface has one A_3, one A_7 and one D_5^0 singularity. In this case, projecting away from E_O contracts the tropes that meet E_O, which are $\hat{T}_1$ and $\hat{T}_2$, into two singularities Q_1 and Q_2 of types A_7 and D_5 respectively, whose coordinates are given by

$$Q_i = [1 : \alpha_i : \alpha_i^2 : \beta_i].$$

From a computation of the Tjurina number of Q_2, we deduce that this singular point has to be of type D_5^0. The remaining singularity $Q_O = [0 : 0 : 0 : 1]$, is of type A_3 as it is a contraction of E_O and two other lines. Similarly to the ordinary case, all the singularities lie in the same plane, which in this case is given by the equation

$$\alpha_1\alpha_2\overline{b}_1 + (\alpha_1 + \alpha_2)\overline{b}_2 + \overline{b}_3 = 0.$$

The intersections of this plane with the Weddle surface are three lines intersecting the three singular points.

The line that has multiplicity two also happens to be the image under the projection from Y to the Weddle surface of the line E_{12}:

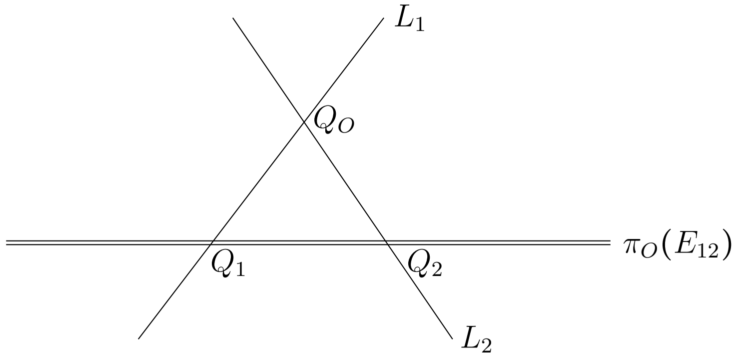

Figure 4.10: Intersections of lines inside the Weddle surface in the almost ordinary case.

Consider now the blow-up of the curve E_O, $\varphi_O : \mathrm{Bl}_{E_O}(Y) \to Y$. Then, the pullbacks of the exceptional lines and tropes of Y are given by

$$\varphi_O^*(E_O) = E_O \times Q_O, \qquad \varphi_O^*(E_{12}) = E_{12} \times \pi_O(E_{12}),$$
$$\varphi_O^*(\hat{T}_1) = \hat{T}_1 \times Q_1, \qquad \varphi_O^*(\hat{T}_2) = \hat{T}_2 \times Q_2.$$

Since we are blowing-up E_O, which contained a singular point of type A_3 corresponding to the intersection of E_O with $\hat{T}_1$, one of the exceptional curves gets blown-up, so that

$$\varphi_O^{-1}(E_O \cap \hat{T}_1) = (E_O \cap \hat{T}_1) \times L_1,$$

and the A_3 singularity becomes an A_2 singularity in the point $(E_O \cap \hat{T}_1) \times Q_O$. Likewise, the singular point of type D_4^1 corresponding to the intersection of E_O with $\hat{T}_2$ gets blown up into the line

$$\varphi_O^{-1}(E_O \cap \hat{T}_2) = (E_O \cap \hat{T}_2) \times L_2,$$

and the D_4^1 singularity becomes an A_3 singularity in the point $(E_O \cap \hat{T}_1) \times Q_2$. None of the other singular points are resolved and we have that their preimages under the blow-up are

$$\varphi_O^{-1}(E_{12} \cap \hat{T}_1) = (E_{12} \cap \hat{T}_1) \times Q_1,$$
$$\varphi_O^{-1}(E_{12} \cap \hat{T}_2) = (E_{12} \cap \hat{T}_2) \times Q_2.$$

Therefore, $\mathrm{Bl}_{E_O}(Y)$ has one A_2, two A_3 and one D_4^1 singularity. One can also understand $\mathrm{Bl}_{E_O}(Y)$ as a blow-up of the Weddle surface that blows up one of the curves in the tail of the A_3 singularity Q_O, the central curve of the A_7 singularity Q_1 (so it splits into two A_3) and the exceptional curve in the tail of the D_5^0 singularity Q_2 (so it becomes a D_4^1).

The blow-up process is described by the following diagram:

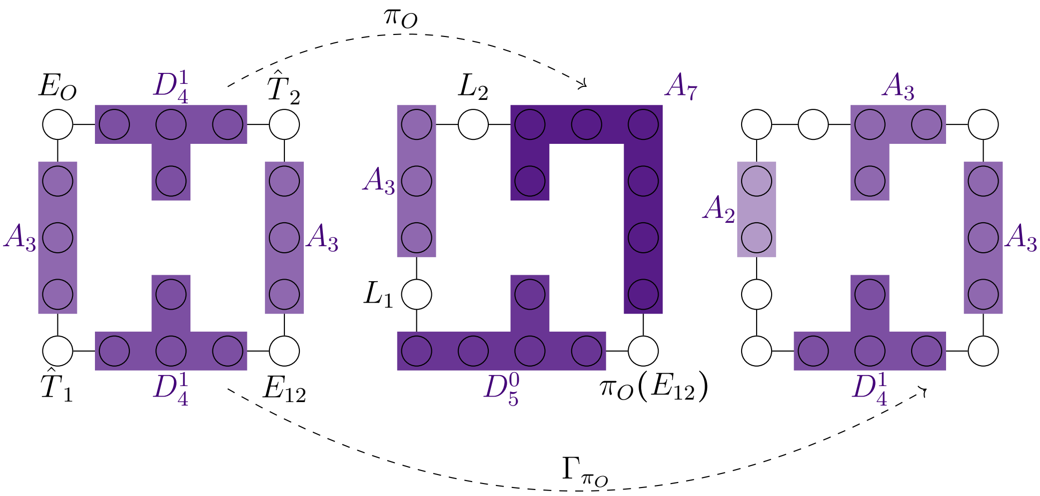

Figure 4.11: Blow-ups of the Kummer surface in the almost ordinary case.

The action of $(\mathbb{Z}/2\mathbb{Z})^2$ on Y allows us to map E_{12}, $\hat{T}_1$ or $\hat{T}_2$ to E_O, so blowing-up any of those lines will produce the same configuration of singularities as blowing-up E_O. Replicating the proof of Proposition 4.6.1, if we consider π_{12} to be the map $Y \dashrightarrow \mathbb{P}^3$ whose image is the four linear polynomials on $\{\bar{b}_1, \ldots, \bar{b}_6\}$ defining the equations of E_{12}, we can construct a morphism $\phi : Y \dashrightarrow (\mathbb{P}^3)^2$ such that the Zariski closure of the image of Γ_ϕ is the blow-up scheme $\mathrm{Bl}_{E_O \cup E_{12}}(Y)$. The singular points of $\mathrm{Bl}_{E_O \cup E_{12}}(Y)$ are then two A_2 and two A_3 singularities. Therefore, in the almost ordinary case it does not suffice to blow-up all the exceptional lines on Y to obtain a smooth model.

As before, we could study this model of desingularisation from specialisation from characteristic zero to characteristic two. Moreover, as we already have a description of how we can specialise from characteristic zero to the ordinary case in characteristic two, it would be enough to see how the ordinary case specialises to the almost ordinary case.

As previously described in Subsection 4.5.2, we can go from the ordinary case to the almost ordinary case by setting one of the three roots of $g(x)$, e.g. α_3 to be equal to α_2. Then, in $\overline{Y}$, this implied that $\{E_O, E_{23}\}$ specialised to E_O, $\{E_{12}, E_{13}\}$ specialised to E_{12}, $\{\hat{T}_1, \hat{T}_{123}\}$ specialised to $\hat{T}_1$ and $\{\hat{T}_2, \hat{T}_3\}$ specialised to $\hat{T}_2$. In the Weddle surface $\overline{W}$ associated to $\overline{Y}$:

- The singular points $\{Q_O, Q_{23}\}$ specialise to Q_O, $\{Q_1, Q_{12}, Q_{13}\}$ specialise to Q_1, and $\{Q_2, Q_3\}$ specialise to Q_2.
- The two lines $\{L_1, \pi_O(\hat{T}_{123})\}$ specialise to L_1, the three lines $\{L_2, L_3, \pi_O(E_{23})\}$ specialise to L_2 and the two lines $\{\pi_O(E_{12}), \pi_O(E_{13})\}$ specialise to $\pi_O(E_{12})$.

As a result, we can see that the specialisation to the almost ordinary case completely breaks the nice symmetries that we had in the ordinary case.

For instance, we see that sometimes we have 2-to-1 reduction and sometimes 3-to-1 reduction, and that there are instances of tropes reducing to lines that are not tropes.

From this, we can see that the description of how the Picard lattice of a smooth Kummer surface with almost ordinary reduction at two reduces is less straight-forward than in the ordinary case. In this case, the sixteen lines E_{ij} that generated the sublattice $\oplus_{i=1}^{16} A_1$ cannot possibly reduce to the generators of the sublattice $D_8 \oplus D_8$ in the reduced surface, and instead, we have that this sublattice must come from $\mathbb{Q}$-linear combinations of the E_{ij} or, alternatively, linear combinations which involve sums of the E_{ij} and the tropes.

4.6.3 The supersingular case

This case is completely different than the previous two. Projecting away from E_O, this time we obtain quite a different outcome as, unlike in the other two cases, the associated Weddle surface no longer has isolated singularities, but instead, it has a singular line L which is defined by the equation:

$$\alpha_1 \overline{b}_1 + \overline{b}_2 = \alpha_1^2 \overline{b}_1 + \overline{b}_3 = 0.$$

The trope $\hat{T}_1$ then gets contracted to the point

$$Q_1 = [1 : \alpha_1 : \alpha_1^2 : \beta_1].$$

Finally, $\mathrm{Bl}_{E_O}(Y)$ blows up the singular point P of Y into a singular line which corresponds to $P \times L$.

4.7 Kummer surfaces that have everywhere good reduction over a quadratic field

Let F be a number field and v a non-Archimedean place of F such that $K = F_v$ is a complete discretely valued field with ring of integers $\mathcal{O}_K$ and residue field k. A variety X/F is said to have **good reduction at** v if there exists a scheme or algebraic space $\mathcal{X}$ smooth and proper over $\mathcal{O}_K$ with generic fibre $\mathcal{X}_K \cong X$.

We will say that X/F has **potentially good reduction at** $\boldsymbol{v}$, if there exists a finite field extension L/F such that for all places w lying above v, X/L has good reduction at w. A variety X/F is said to have **everywhere good reduction** if it has good reduction at every non-Archimedean place.

There is a well-known result of Fontaine [Fon85] (see also Abrashkin [Abr88]) which asserts that there does not exist any abelian scheme over $\mathbb{Z}$ and, as a consequence of this, there cannot exist abelian varieties defined over $\mathbb{Q}$ with everywhere good reduction.

In a similar fashion, a lesser-known result, also due to Abrashkin [Abr90] and Fontaine [Fon91] independently, shows that there cannot exist K3 surfaces defined over the rationals that have everywhere good reduction.

Since Tate provided in the late sixties one of the first examples of elliptic curves with good reduction everywhere, the curve $E/\mathbb{Q}(\sqrt{29})$ defined as

$$\mathcal{E} : y^2 + xy + \left(\tfrac{5+\sqrt{29}}{2}\right)^2 y = x^3,$$

many different techniques and methods have been developed in order to find elliptic curves with everywhere good reduction over number fields. In the case of abelian surfaces, it is relevant the work of Dembélé and Kumar [DK16], Dembélé [Dem21], and Dąbrowski and Sadek [DS21] who all found explicit examples defined over quadratic fields of genus two curves whose Jacobians have everywhere good reduction over a quadratic number field.

After seeing that the question has a positive answer for abelian surfaces, one would naturally ask if it is then possible to find examples of K3 surfaces with everywhere good reduction over a number field. This is indeed the case for Kummer surfaces, where we can find a scheme model with everywhere good reduction, as a consequence of the following.

Let A be an abelian surface over a number field K and let v be an non-Archimedean place.

- If v does not lie above two, then the Kummer surface associated to A has good reduction at v if and only if there exists a quadratic twist A^χ of A such that A^χ has good reduction. This is a consequence of the work of Matsumoto [Mat15] and Overkamp [Ove21].
- If v lies above two, then the Kummer surface associated to A has potentially good reduction if A has good reduction at v. This is a consequence of Lazda and Skorobogatov [LS23] in the ordinary and almost ordinary case and Matsumoto [Mat23] in the supersingular case.

Starting with an abelian surface with everywhere good reduction, these results show that over possibly a field extension, its associated Kummer surface has everywhere good reduction. The goal of this section is to show that it is possible to explicitly construct an example of a Kummer surface with everywhere good reduction over a quadratic number field.

Theorem 4.7.1. *Let $F = \mathbb{Q}(\sqrt{353})$, let $\omega = \frac{1+\sqrt{353}}{2}$ and let*

$$\mathcal{C} : y^2 + g(x)y = f(x)$$

where

$$\begin{aligned} g(x) &= (\omega+1)x^3 + x^2 + \omega x + 1, \\ f(x) &= (-15\omega + 149)x^6 - (1119\omega + 9948)x^5 - (36545\omega + 325409)x^4 \\ &\quad - (363632\omega + 5659370)x^3 - (622714\omega + 5538975)x^2 \\ &\quad - (3284000\omega + 288867915)x - 70532813\omega - 627353458. \end{aligned}$$

Then, the Kummer surface associated to $\mathrm{Jac}(\mathcal{C})$ has everywhere good reduction over F.

Proof. This curve was found by Dembélé [Dem21, Theorem 6.2]. One can check that the discriminant of $\mathcal{C}$ is $-\epsilon^4$, where ϵ is the fundamental unit of F, and therefore $\mathrm{Jac}(\mathcal{C})$ has everywhere good reduction. By the previously mentioned results, its associated Kummer surface has good reduction at all non-Archimedean places not lying above two. Therefore, we only need to prove that the Kummer surface also has good reduction at the places lying above two. In order to do that, we will apply a criterion developed by Lazda and Skorobogatov [LS23, Theorem 2].

4.7.1 A criterion for good reduction

Let $A = \mathrm{Jac}(\mathcal{C})$ be an abelian surface with good (not supersingular) reduction at two, let K be a discretely valued field with perfect residue field k of characteristic two, and let $\mathcal{A}/\mathcal{O}_K$ be the Néron model of A/K, which is an abelian scheme with generic fiber $\mathcal{A}_K \cong A$. Let us fix an algebraic closure $\overline{K}$ of K, with residue field $\overline{k}$, and let Γ_K denote the Galois group of $\overline{K}/K$. Then, we have the exact sequence of Γ_K-modules:

$$0 \longrightarrow \mathcal{A}[2]^\circ(\overline{K}) \longrightarrow \mathcal{A}[2](\overline{K}) \longrightarrow \mathcal{A}[2](\overline{k}) \longrightarrow 0 \tag{4.7.1}$$

where $\mathcal{A}[2]^\circ$ is the connected component of the identity of the 2-torsion subscheme $\mathcal{A}[2] \subseteq \mathcal{A}$.

Theorem 4.7.2 ([LS23, Theorem 6.3]). *If A has ordinary reduction, the Kummer surface associated to A has good reduction over K if and only if the exact sequence (4.7.1) of Γ_K-modules split. If A has almost ordinary reduction, the Kummer surface associated to A has good reduction over K if and only if the Γ_K-module $\mathcal{A}[2](\overline{K})$ is trivial. Moreover, in both cases the Kummer surface has good reduction with a scheme model.*

4.7.2 The proof of theorem 4.7.1

As the curve $\mathcal{C}$ has ordinary reduction at two, we will apply the first part of the theorem. Let the K in the previous theorem be the completion of $F = \mathbb{Q}(\sqrt{353})$ at two. As 353 is 1 modulo 8, we can easily check that 353 is a square in $\mathbb{Q}_2$, and so, $K = \mathbb{Q}_2$.

Then, $\mathcal{O}_K = \mathbb{Z}_2$ and we deduce that $k = \mathbb{F}_2$. Furthermore, by computing the 2-adic expansion, we can see that ω reduces to zero modulo two and therefore the reduction of $\mathcal{C}$ modulo two can be shown to have the equation

$$y^2 + (x^3 + x^2 + 1)y = x^6 + x^2 + x.$$

As explained in section 4.5, the decomposition of $g(x)$ over k determines the number of 2-torsion points defined over k. As in this case $g(x)$ is irreducible over $\mathbb{F}_2$, $\mathcal{A}[2](k)$ is trivial and $\mathcal{A}[2](\ell) = (\mathbb{Z}/2\mathbb{Z})^2$ if and only if $\ell \supseteq \mathbb{F}_8 = \mathbb{F}_2(\overline{\gamma})$, where $\overline{\gamma}^3 + \overline{\gamma}^2 + 1 = 0$. The 2-torsion points are of the form $\{\overline{P}_O, \overline{P}_{12}, \overline{P}_{13}, \overline{P}_{23}\}$ (as described in section 4.5) where we take $\alpha_1 = \overline{\gamma}$, $\alpha_2 = \overline{\gamma}^2$ and $\alpha_3 = \overline{\gamma}^2 + \overline{\gamma} + 1$.

Therefore, as a Γ_K-module $\mathcal{A}[2](\overline{k})$ only admits a cyclic action of order three permuting its non-trivial elements corresponding to the action of Frobenius in $\mathbb{F}_8$. On the other hand, the number of 2-torsion points defined over K is determined by the decomposition of $f(x) + \frac{1}{4}g(x)^2$ into irreducible polynomials over K, and using Magma, we can easily check that

$$f(x) + \tfrac{1}{4}g(x)^2 = \tfrac{1}{4}q_1(x)q_2(x),$$

where q_1 and q_2 are the following irreducible polynomials over $\mathbb{Q}_2$

$$\begin{aligned} q_1(x) &= x^3 + (2088841801 + O(2^{32}))x^2 + (1097586240 + O(2^{32}))x + 553607353 + O(2^{32}),\\ q_2(x) &= x^3 + (1373013921 + O(2^{32}))x^2 - (1548938988 + O(2^{32}))x - 856394843 + O(2^{32}). \end{aligned}$$

As a matter of fact, this decomposition is induced by the fact that over F,

$$f(x) + \tfrac{1}{4}g(x)^2 = -\tfrac{3}{4}(19\omega + 169)q_1(x)q_2(x)$$

where

$$\begin{aligned} q_1(x) &= x^3 + \tfrac{1}{3}(12\omega - 5)x^2 + \tfrac{1}{12}(11\omega + 5640)x + \tfrac{1}{12}(2507\omega - 588),\\ q_2(x) &= x^3 + (4\omega + 1)x^2 + (8\omega + 468)x + 211\omega + 365. \end{aligned}$$

As $f(x) + \frac{1}{4}g(x)^2$ decomposes into two cubic polynomials, $|\mathcal{A}[2](K)| = 1$ and as $\mathcal{A}[2](K) \neq \mathcal{A}[2](\overline{K})$, we deduce that there are elements of Γ_K acting non-trivially on $\mathcal{A}[2](\overline{K})$.

Let L be the unique unramified extension of degree three of $\mathbb{Q}_2$ which, without any loss of generality, we can consider it to be $\mathbb{Q}_2(\gamma)$ where $(\omega + 1)\gamma^3 + \gamma^2 + \omega\gamma + \omega + 1 = 0$. Then, over L, we have that

$$f(x) + \tfrac{1}{4}g(x)^2 = \tfrac{1}{4}h_1(x)h_2(x)h_3(x)h_4(x)h_5(x)h_6(x),$$

where

$$\begin{aligned}
h_1(x) &= x - 406904280\gamma^2 + 435522127\gamma - 1230442616 + O(2^{32}),\\
h_2(x) &= x + 394057577\gamma^2 - 1606502354\gamma + 490223466 + O(2^{32}),\\
h_3(x) &= x - 1060895121\gamma^2 - 976503421\gamma + 681577303 + O(2^{32}),\\
h_4(x) &= x + 1307484884\gamma^2 + 1755128143\gamma - 56114964 + O(2^{32}),\\
h_5(x) &= x + 914512901\gamma^2 + 842339586\gamma - 1344868422 + O(2^{32}),\\
h_6(x) &= x - 1148255961\gamma^2 - 449984081\gamma + 626513659 + O(2^{32}),
\end{aligned}$$

and $q_1(x) = h_1(x)h_2(x)h_3(x)$ and $q_2(x) = h_4(x)h_5(x)h_6(x)$. Let r_i denote the root of h_i, and let P_{ij} be the 2-torsion point associated to r_i and r_j. As the polynomial completely splits over L, $\mathcal{A}[2](L) = \mathcal{A}[2](\overline{L})$ and, therefore, $\mathcal{A}[2](\overline{L})$ is trivial as a Γ_L-module. We can therefore check that the only non-trivial actions of Γ_K in $\mathcal{A}[2](K)$ are the ones induced by $\mathrm{Gal}(L/K) \cong C_3$ which permute the roots of q_1 and q_2.

As L is the maximal unramified extension of degree three of $\mathbb{Q}_2$, the action of Γ_K on $\mathcal{A}[2](\overline{k})$ is also by the group C_3 and it acts in a way that is compatible with the action on $\mathcal{A}[2](\overline{k})$. More precisely, let $\varsigma \in S_6$ given in the cycle notation by $\varsigma = (123)(456)$, and let τ_ς be the action of Γ_K induced in $\mathcal{A}[2](\overline{K})$ by $\tau_\varsigma(P_{ij}) = P_{\varsigma(i)\varsigma(j)}$. Then, τ_ς acts on $\mathcal{A}[2](\overline{k})$ by permuting cyclically the roots of $g(x)$ and the short exact sequence

$$0 \longrightarrow \mathcal{A}[2]^\circ(\overline{K}) \longrightarrow \mathcal{A}[2](\overline{K}) \xrightarrow{f} \mathcal{A}[2](\overline{k}) \longrightarrow 0$$

splits as we can easily construct sections of it, for instance, by defining

$$\sigma(P) = \begin{cases} P_O & \text{if } P = \overline{P}_O \\ P_{12} & \text{if } P = \overline{P}_{12} \\ P_{13} & \text{if } P = \overline{P}_{13} \\ P_{23} & \text{if } P = \overline{P}_{23} \end{cases}$$

as $\langle P_{12}, P_{13}\rangle = (\mathbb{Z}/2\mathbb{Z})^2 \subset \mathcal{A}[2](\overline{K})$. It can be checked that r_1, r_2 and r_3 reduce to α_1, α_2 and α_3 respectively so $f \circ \sigma = id$. Notice that there are multiple acceptable different ways to construct sections, such as considering the images of $\{\overline{P}_O, \overline{P}_{12}, \overline{P}_{13}, \overline{P}_{23}\}$ to be $\{P_O, P_{45}, P_{46}, P_{56}\}$, $\{P_O, P_{15}, P_{34}, P_{26}\}$, or $\{P_O, P_{15}, P_{46}, P_{23}\}$, for instance. As a matter of fact, there are sixteen possible sections that we can take, as there are four possible images for $\overline{P}_{12}$, four possible images for $\overline{P}_{13}$, and once we fix $\sigma(\overline{P}_{12})$ and $\sigma(\overline{P}_{13})$, then $\sigma(\overline{P}_{23})$ must be defined to be $\sigma(\overline{P}_{12}) + \sigma(\overline{P}_{13})$. □

This is where we can draw a connection with the previous sections of the chapter. By choosing a section of the short sequence, we are choosing a set of four 2-torsion points $\{P_O, P_{12}, P_{13}, P_{23}\}$ with the same Galois action as the 2-torsion over the residue field. In the model of the Kummer surface as an intersection of three quadrics in $\mathbb{P}^5$, the exceptional lines $\{E_O, E_{12}, E_{13}, E_{23}\}$ are defined over the same extension of $\mathbb{Q}_2$ as the torsion points they come from, and their union is defined over $\mathbb{Z}_2$, as the ideal defining this variety only depends on the coefficients of the polynomial q_1.

Due to the Galois action over K being compatible with the Galois action of the reduction, we deduce that the ideal corresponding to the union of these four lines must reduce to the ideal of the four exceptional lines associated to the 2-torsion points over the residue field $\mathbb{F}_2$. If we consider the blow-up of the four lines on Y, we would therefore obtain a smooth model of the Kummer surface defined over $\mathbb{Z}_2$ whose reduction would be the blow-up of the four exceptional lines over $\mathbb{F}_2$ which, as we have seen, resolves all twelve singular points.

In this example, we did not need to take any field extension to obtain good reduction of the Kummer surface at two. This is not generally the case, as we can see when we analyse the other examples in the articles, where we only obtain potential good reduction at the primes above two and we need to take field extensions to achieve good reduction.

In the following table, we can see all the examples of curves $\mathcal{C}$ with ordinary reduction at two and everywhere good reduction over the field $\mathbb{Q}(\omega)$, the first six from the article of Dembélé and Kumar [DK16], and the last two from the article of Dembélé [Dem21]. In the last column, we can find the degree of the minimal extension of $\mathbb{Q}_2(\omega)$ where $\mathrm{Kum}(\mathcal{C})$ acquires good reduction at two. All the computations can be found in the file `Everywhere good reduction.m`.

$g(x)$	$f(x)$	ω	d
$\omega x^3+\omega x^2+\omega+1$	$-4x^6+(\omega-17)x^5+(12\omega-27)x^4+(5\omega-122)x^3$ $+(45\omega-25)x^2+(-9\omega-137)x+14\omega+9$	$\frac{1+\sqrt{53}}{2}$	2
x^3+x+1	$(\omega-5)x^6+(3\omega-14)x^5+(3\omega-19)x^4$ $+(4\omega-3)x^3-(3\omega+16)x^2+(3\omega+11)x-(\omega+4)$	$\frac{1+\sqrt{73}}{2}$	4
$\omega\,(x^3+1)$	$-2(4414\omega+43089)x^6+(31147\omega+303963)x^5$ $-10(4522\omega+44133)x^4+2(17290\omega+168687)x^3$ $-18(816\omega+7967)x^2+27(122\omega+1189)x-(304\omega+3003)$	$\frac{1+\sqrt{421}}{2}$	2
x^3+x^2+1	$-2x^6+(-3\omega+1)x^5-219x^4+(-83\omega+41)x^3-1806x^2$ $+(-204\omega+102)x-977$	$\frac{1+\sqrt{409}}{2}$	4
x^3+x+1	$-134x^6-(146\omega-73)x^5-13427x^4-(3255\omega-1627)x^3$ $-89746x^2-(6523\omega-3261)x-39941$	$\frac{1+\sqrt{809}}{2}$	4
x^3+x+1	$23x^6+(90\omega-45)x^5+33601x^4+(28707\omega-14354)x^3$ $+3192149x^2+(811953\omega-405977)x+19904990$	$\frac{1+\sqrt{929}}{2}$	4
$\omega x^3+x^2+(\omega+1)x+1$	$(13\omega+77)x^6+(503\omega+6772)x^5+(1504\omega+131460)x^4$ $+(16882\omega+1727293)x^3+(116734\omega+10787410)x^2$ $+(398570\omega+40121781)x+611123\omega+58505073$	$\frac{1+\sqrt{421}}{2}$	4
$x^3+\omega x^2+(\omega+1)x+\omega+1$	$(14154412\omega+275745514)x^6-(489014393\omega+9526607332)x^5$ $+(7039395048\omega+137136152764)x^4-54043428224\omega x^3$ $-1052833060832x^3+(233382395752\omega+4546578743807)x^2$ $-(537510739916\omega+10471376373574)x+515810377784\omega$ $+10048626384323$	$\frac{1+\sqrt{1597}}{2}$	4

Table 4.5: Examples of curves with everywhere good reduction and ordinary reduction at two.

To see why for some examples of surfaces we need to consider a field extension in order to acquire good reduction at two, let us look for instance at the third example of the table.

Here, $K = \mathbb{Q}_2(\sqrt{421})$, $\mathcal{O}_K = \mathbb{Z}_2[\omega]$ and as the minimal polynomial of ω is $x^2-x-105$, which is irreducible modulo two, we deduce that $k = \mathbb{F}_2(\overline{\omega}) = \mathbb{F}_4$. Then, the reduction of $\mathcal{C}$ modulo two can be shown to have the equation

$$y^2 + \overline{\omega}(x^3+1) = (1+\overline{\omega})x^5 + x + 1.$$

Therefore, $g(x)$ completely splits over k

$$g(x) = \overline{\omega}(x^3+1) = (\overline{\omega}x+1)(x+1)(x+\overline{\omega}),$$

and as $\mathcal{A}[2](\overline{k}) = \mathcal{A}[2](k)$, we deduce that $\mathcal{A}[2](\overline{k})$ is trivial as a Γ_K-module. However,

$$f(x) + \tfrac{1}{4}g(x)^2 = \tfrac{1}{4}h_1(x)h_2(x)q_3(x)q_4(x),$$

where

$$\begin{aligned}
&h_1(x) = x + 1312351119 - 2028179001\omega + O(2^{32}),\\
&h_2(x) = x - 1300818437 - 1345357737\omega + O(2^{32}),\\
&q_3(x) = x^2 + (1256541238 + 188416644\omega + O(2^{32}))\,x + (1294873809 - 1495287772\omega + O(2^{32})),\\
&q_4(x) = x^2 + (-1426178004 - 209135522\omega + O(2^{32}))\,x + (-1663860799 + 724531893\omega + O(2^{32})),
\end{aligned}$$

are all irreducible polynomials over K. This implies that $\mathcal{A}[2](\overline{K})$ is not trivial as a Γ_K-module, as there are non-trivial K-automorphisms acting on the Weierstrass points, and therefore the 2-torsion. For instance, we have an action of order two permuting the two roots of q_3.

We can check that the only submodule of $\mathcal{A}[2](\overline{K})$ that is trivial as a Γ_K-module is $\mathcal{A}[2](K)$, which, as a group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ by what we have described in section 4.2. However, this submodule is precisely $\mathcal{A}[2]^\circ(\overline{K})$, and the image of this group in $\mathcal{A}[2](\overline{k})$ is trivial.

As a consequence, one cannot find a section of the exact sequence (4.7.1), as the image of any section would have to be trivial as a Γ_K-module, and we deduce that $\mathrm{Kum}(A)$ does not have good reduction over $\mathbb{Q}_2(\sqrt{421})$. However, if we consider the ramified extension $L = \mathbb{Q}_2(\sqrt{421}, i)$, then, over that extension, the polynomial $f(x) + \frac{1}{4}g(x)^2$ completely splits. Thus, $\mathcal{A}[2](\overline{L})$ becomes trivial as a Γ_L-module, and we can easily construct sections as in the previous example.

Through a similar reasoning, we can argue in the other seven examples which field extension we need to take, and what its degree is.

In the first example, $f(x) + \frac{1}{4}g(x)^2$ decomposes into the product of a quadratic and a quartic polynomial over $K = \mathbb{Q}_2(\omega)$ and the splitting field has Galois group C_2^3. Over the residue field $k = \mathbb{F}_4$, $g(x)$ decomposes into a linear and a quadratic factor, therefore the action of Γ_K on $\mathcal{A}[2](\overline{K})$ is by the group C_2. We checked that there are two possible quadratic extensions of K compatible with the Galois action where the sequence (4.7.1) splits, namely, the two ramified extensions that split the quartic factor of $f(x) + \frac{1}{4}g(x)^2$. Each of these gives rise to eight possible sections that would split the sequence, so that in total over the splitting field we would have the sixteen possible sections that we described earlier.

In the rest of the examples, we always have that $f(x) + \frac{1}{4}g(x)^2$ is irreducible over K and the splitting field has A_4 as its Galois group. Furthermore, over the residue field, $g(x)$ is also irreducible so its Galois group is C_3. From the Sylow theorems, we deduce that there are four Sylow 3-subgroups, which have index four in A_4, and from the Galois correspondence we deduce that these must correspond to four field extensions of K of degree four. Over any of these extensions, $f(x) + \frac{1}{4}g(x)^2$ splits into two cubic polynomials and we can construct four sections splitting the sequence (4.7.1). These extensions were generally easy to find, with the exception of the last two examples where, in order to find the extensions over which the sequences split, we had to explicitly find the fixed field by the Sylow 3-subgroups.

4.7.3 Kummer surfaces with everywhere good reduction and almost ordinary reduction at two

A natural question remains unanswered, which is whether it is possible to construct a Kummer surface with everywhere good reduction and almost ordinary reduction at two. The answer in this case is also affirmative, but no examples have been found where the good reduction is achieved over a quadratic number field. In this case, as we saw from theorem 4.7.2, good reduction at two is obtained over the field K where the sequence (4.7.1) is of trivial Γ_K-modules. Applying the same reasoning as before, we can see that this field extension K must be the splitting field of $f(x) + \frac{1}{4}g(x)^2$.

There are only two examples in Dembélé and Kumar's article of abelian surfaces with everywhere good reduction that have good almost ordinary reduction at two:

$g(x)$	$f(x)$	ω	d
$-x-\omega$	$2x^6 + (-2\omega+7)x^5 + (-5\omega+47)x^4 + (-12\omega+85)x^3$ $+(-13\omega+97)x^2 + (-8\omega+56)x - 2w + 1$	$\frac{1+\sqrt{193}}{2}$	12
$x+1$	$-2x^6 - (2\omega-1)x^5 - 45x^4 - 4(2\omega-1)x^3 - 31x^2$ $+(\omega-1)x + 9$	$\frac{1+\sqrt{233}}{2}$	12

Table 4.6: Examples of curves with everywhere good reduction and almost ordinary reduction at 2.

In both of this cases, we can check that the minimal extension where $f(x) + \frac{1}{4}g(x)^2$ completely splits is the degree twelve extension $\mathbb{Q}_2(\sqrt{5}, \sqrt[3]{1+i})$, whose Galois group is the dihedral group of order twelve. The calculations are available in the file `Everywhere good reduction.m` as well.

One can easily check that this field extension decomposes in a degree two unramified part $\mathbb{Q}_2(\sqrt{5})$, and a degree six completely ramified part given by $\mathbb{Q}_2(\sqrt[3]{1+i})$. Therefore, if we set $K = \mathbb{Q}_2(\sqrt[3]{1+i})$, we find that the Jacobians of any of the two previous examples are abelian surfaces with good, almost ordinary reduction at two, such that $\mathcal{A}[2](\overline{K})$ are unramified but non-trivial as a Γ_K-module.

Regarding other possible examples, Dąbrowski and Sadek [DS21] computed a family of genus two curves with everywhere good reduction and almost ordinary reduction at two. We computed 400 examples of their family and checked that in none of them, the associated Kummer surface has good reduction over the base field.

4.7.4 Other examples

The rest of the examples of the list of abelian surfaces with everywhere good reduction over a quadratic field have supersingular reduction at two. By the result by Matsumoto [Mat23, Theorem 1.2], there must be a field extension over which the Kummer surface acquires good reduction at two, but unfortunately, his result does not allow us to explicitly compute what this extension is in examples.

However, very recently, there has been new progress in the problem of computing examples of Kummer surfaces with everywhere good reduction over a number field when the Kummer surface has supersingular reduction at 2. Schröer has constructed examples of Kummer surfaces of products of two elliptic curves over the quadratic number fields $\mathbb{Q}(\sqrt{7})$, $\mathbb{Q}(\sqrt{41})$ and $\mathbb{Q}(\sqrt{65})$, and an infinite family of cubic fields with Galois group S_3 [Sch25].

5 Intersections of the automorphism and the Ekedahl-Oort strata in M_2

This chapter is based on the paper *Intersections of the automorphism and the Ekedahl-Oort strata in M_2*.

5.1 Introduction

In chapter 2, we saw that the number and type of singular points of a generalised Kummer surface associated to the quotient of an abelian surface A by the action of a group G was determined by the characteristic p of the base field, and, if p divided the order of G, the p-rank of A.

In the case where $A = \mathrm{Jac}(\mathcal{C})$ for some curve $\mathcal{C}$, we can make a connection between the automorphisms of $\mathcal{C}$ and the automorphisms of A preserving its principal polarisation as a Jacobian variety (which, as we saw in section 4.2, comes from considering a multiple of a theta divisor). In the next chapter, we will study which generalised Kummer surfaces arise as quotients of $\mathrm{Jac}(\mathcal{C})$ by a group action preserving the polarisation. But before we do that, in this chapter we will answer the following question:

Given a prime p, $0 \leq f, a \leq 2$, and a group G, is there a genus two curve over a field of characteristic p that has automorphism group G and whose Jacobian variety has p-rank f and a-rank a? If the answer is affirmative, we would also like to obtain more information about the properties of the set of curves satisfying these conditions.

One thing that we can do is to study what the locus of the curves satisfying these properties looks like inside the coarse moduli space of genus two curves M_2. The reason why this approach is convenient is because an explicit description of this moduli space is known, and we can perform computations in it using mathematical software, using Magma in our case [BCP97]. The code is available in this repository.

In section 5.2, we review the construction of an explicit model for the coarse moduli space of genus two curves using the Igusa invariants.

In section 5.3, we survey what the possible automorphism groups of a genus two curve are in every characteristic.

This section is of independent interest, as we provide a detailed description of the geometry of the automorphism strata and compute universal models for the curves in each stratum, extending the models described by Cardona and Quer [CQ07] to work uniformly in every characteristic. In the case of curves with automorphism group C_2^2, we prove that this stratum can be related to a moduli space parametrising pairs of elliptic curves with level 2 structures.

In section 5.4, we recall the description of the Ekedahl-Oort strata of the moduli space of abelian surfaces, and explain how to compute these strata from the description of the Hasse-Witt matrix.

Finally, in section 5.5, we present the main results of the chapter, which are tables containing the dimensions and the number of irreducible components of the strata corresponding to the intersections of the automorphism and Ekedahl-Oort strata.

5.2 The coarse moduli space of genus two curves

Let M_2 denote the coarse moduli space of genus two curves. Assuming that we are working over a field k of characteristic not two, then, over the algebraic closure of k every genus two curve can be written in the form

$$y^2 = \prod_{i=1}^{6}(x-\lambda_i), \tag{5.2.1}$$

for some $\lambda_i \in \overline{k}$ which are all distinct. In 1960, Igusa [Igu60] computed a set of five functions now known as **Igusa invariants** $\{J_2, J_4, J_6, J_8, J_{10}\}$ that determine a genus two curve up to isomorphism. These invariants are defined as symmetric functions depending on the λ_i in the following way:

For any permutation $\sigma \in S_6$, we define $(ij)_\sigma := (\lambda_{\sigma(i)} - \lambda_{\sigma(j)})$. Let s_1, s_2, s_3 be the following symmetric polynomials:

$$\begin{aligned}
s_1 &= \sum_{\sigma\in S_6} (12)_\sigma^2(34)_\sigma^2(56)_\sigma^2,\\
s_2 &= \sum_{\sigma\in S_6} (12)_\sigma^2(13)_\sigma^2(23)_\sigma^2(45)_\sigma^2(46)_\sigma^2(56)_\sigma^2,\\
s_3 &= \sum_{\sigma\in S_6} (12)_\sigma^2(13)_\sigma^2(14)_\sigma^2(23)_\sigma^2(25)_\sigma^2(36)_\sigma^2(45)_\sigma^2(46)_\sigma^2(56)_\sigma^2,\\
s_5 &= \prod_{1\le i<j\le 6} (ij)_{\mathrm{id}}^2 = \operatorname{disc}\Big(\prod_{i=1}^{6}(x-\lambda_i)\Big).
\end{aligned}$$

Then, the Igusa invariants are defined by

$$\begin{aligned}
J_2 &= \frac{1}{2^3 3} s_1, \\
J_4 &= \frac{1}{2^9 3^3} s_1^2 - \frac{1}{3^3} s_2, \\
J_6 &= \frac{1}{2^{13} 3^6} s_1^3 + \frac{5}{2^4 3^6} s_1 s_2 - \frac{2^4}{3^3} s_3, \\
J_8 &= \frac{1}{2^{20} 3^7} s_1^4 + \frac{13}{2^{10} 3^7} s_1^2 s_2 - \frac{1}{2\, 3^4} s_1 s_3 - \frac{1}{2^2 3^6} s_2^2, \\
J_{10} &= 2^8 s_5.
\end{aligned}$$

Some sources define these invariants slightly differently, but we have chosen the definition that matches the implementation of the Igusa invariants in Magma [Mes91].

As these only depend on symmetric polynomials of the x-coordinates of the Weierstrass points of the curve, one can easily check that the Igusa invariants can also be expressed in terms of the coefficients of the curve. Therefore, they are defined over the base field of the curve. The definition of these Igusa invariants extends to curves defined over any field of positive characteristic, in particular, over fields of characteristic two where genus two curves cannot be written as in the equation 5.2.1.

These invariants are not algebraically independent, as for any curve we have the relation

$$J_4^2 - J_2 J_6 + 4J_8 = 0.$$

Since our curves are smooth, we can also check that $J_{10} \neq 0$. Let $\mathfrak{X} = \mathbb{V}(J_4^2 - J_2 J_6 + 4J_8)$. We can then define a morphism

$$\begin{aligned}
M_2 &\longrightarrow \mathfrak{X} \subset \mathbb{P}(1,2,3,4,5) \\
\mathcal{C} &\longmapsto [J_2(\mathcal{C}) : J_4(\mathcal{C}) : J_6(\mathcal{C}) : J_8(\mathcal{C}) : J_{10}(\mathcal{C})],
\end{aligned}$$

and the image of M_2 inside $\mathbb{V}(J_4^2 - J_2 J_6 + 4J_8)$ corresponds to the open subvariety $\mathfrak{X}_0 \subset \mathfrak{X}$ given by $J_{10} \neq 0$. As stated before, the Igusa invariants determine a genus two curve up to isomorphism, and so, there is an inverse rational map from $\mathfrak{X}$ to M_2. Furthermore, the Deligne-Mumford compactification $\overline{M}_2$ is isomorphic to a blow-up of $\mathfrak{X}$ [Liu93, Théorème 1].

The method to recover a curve from its Igusa invariants has been described by Mestre [Mes91]. Note that, in general, given a set of Igusa invariants over a non-algebraically closed field k satisfying $J_4^2 - J_2 J_6 + 4J_8 = 0$, it may not be possible to find a curve defined over k with those invariants.

Let $\mathcal{A}_2$ be the coarse moduli space of principally polarised abelian surfaces. We can construct a morphism from M_2 to $\mathcal{A}_2$ that associates to each curve its Jacobian variety.

This map, known as the **Torelli morphism**, is an injective morphism from M_2 to $\mathcal{A}_2$, and its image is dense in $\mathcal{A}_2$.

Since the moduli spaces $\mathcal{A}_g$ were first described, there has been significant interest in understanding their stratifications. These are decompositions of $\mathcal{A}_g$ into locally closed subsets known as **strata**, which correspond to abelian surfaces with specific geometric or arithmetic properties. For $g = 2$, the Torelli morphism provides a means of translating these properties into the language of curves, allowing one to study the corresponding loci in M_2 as preimages of the relevant strata in $\mathcal{A}_2$.

5.3 The automorphism group stratification

Assume we are working over an algebraically closed field. Then, a genus two curve defined over a field of characteristic $p \geq 0$ has one of the following automorphism groups [Igu60]:

- C_2, which happens generically.
- C_2^2.
- D_4, the dihedral group of order 8, if $p \neq 2$.
- C_{10}, if $p \neq 2, 5$.
- D_6, the dihedral group of order 12.
- $C_3 \rtimes D_4$, which has order 24, if $p \neq 2, 3, 5$.
- $\mathrm{GL}_2(\mathbb{F}_3)$, which has order 48, if $p \neq 2, 5$.

If $p = 2$, we have the additional possibilities:

- C_2^5, which has order 32.
- $C_2 \wr C_5$, which has order 160.

If $p = 5$, we also have:

- $\mathrm{SL}_2(\mathbb{F}_5)$, which has order 120.

A consequence of the injectivity of the Torelli morphism is that every automorphism of $\mathrm{Jac}(\mathcal{C})$ preserving the polarisation arises from a unique automorphism of the curve $\mathcal{C}$. Conversely, as every genus two curve is hyperelliptic, one can check that every automorphism of $\mathcal{C}$ comes from a polarised automorphism of $\mathrm{Jac}(\mathcal{C})$.

Therefore, $\mathrm{Aut}(\mathcal{C})$ is isomorphic to the group of automorphisms of $\mathrm{Jac}(\mathcal{C})$ which preserve the polarisation. We will explain this in more detail in the next chapter.

Let G denote one of the previous groups. Then, the subsets

$$W_{\geq G} = \{\mathcal{C} \in M_2(\bar{k}) : \mathrm{Aut}(\mathcal{C}) \geq G\}$$

are closed subschemes of M_2. This can be seen from the fact that over algebraically closed fields, isomorphic curves have isomorphic automorphism groups and that the necessary conditions for a curve to admit a certain automorphism can be described in terms of the vanishing of polynomials involving the coefficients of the curve. The dimensions of the subschemes $W_{\geq G}$ are the following:

				$p \neq 2$	$p \neq 2,5$		$p \neq 2,3,5$	$p = 2$		$p = 5$
G	C_2	C_2^2	D_6	D_4	C_{10}	$\mathrm{GL}_2(\mathbb{F}_3)$	$C_3 \rtimes D_4$	C_2^5	$C_2 \wr C_5$	$\mathrm{SL}_2(\mathbb{F}_5)$
$\dim(W_{\geq G})$	3	2	1	1	0	0	0	1	0	0

Table 5.1: Dimensions of the automorphism strata.

5.3.1 The zero-dimensional strata

Each of the zero-dimensional strata corresponds to a unique curve:

1. Up to isomorphism, in characteristics not two or five, the unique curve with automorphism group C_{10} is

$$y^2 + y = x^5,$$

   which corresponds to the point $[0:0:0:0:1]$ in M_2.

2. Up to isomorphism, in characteristics not two or five, the unique curve with automorphism $\mathrm{GL}_2(\mathbb{F}_3)$ is

$$y^2 = x^5 - x.$$

   In M_2, this curve corresponds to the point $[20:30:-20:-325:64]$.

3. Up to isomorphism, in characteristics not two, three or five, the unique curve with automorphism group $C_3 \rtimes D_4$ is

$$y^2 = x^6 + 1.$$

   In M_2, this curve corresponds to the point $[120:330:-320:-36825:11664]$.

4. Up to isomorphism, the unique curve in characteristic two with automorphism $C_2 \wr C_5$ is

$$y^2 + y = x^5.$$

   In M_2, this curve corresponds to the point $[0:0:0:0:1]$.

5. Up to isomorphism, the unique curve in characteristic five with automorphism $\mathrm{SL}_2(\mathbb{F}_5)$ is

$$y^2 = x^5 - x.$$

In M_2, this curve corresponds to the point $[0:0:0:0:1]$.

5.3.2 Describing the other automorphism strata

The only strata that are left are the ones corresponding to the groups C_2^2, D_4 and D_6 in characteristic not two and C_2^5 in characteristic two. The key idea to explicitly construct these strata is that we can characterise these automorphism groups from the presence of elements of a certain order. More specifically,

- The genus two curves with automorphism group containing C_2^2 are those that admit a non-hyperelliptic automorphism of order two.
- The genus two curves with automorphism group containing D_4 are those that admit an automorphism of order four.
- The genus two curves with automorphism group containing D_6 are those that admit an automorphism of order three.

In characteristic two, the stratum C_2^5 has been described by Igusa [Igu60].

Given any genus two curve, we can always transform it to one of the form

$$y^2 + \sum_{i=0}^{3} g_i x^i y = \sum_{j=0}^{6} f_j x^j.$$

for some $f_i, g_j \in \overline{k}$. In order to construct the strata, we will assume that this general curve admits one of the special automorphisms stated above. In order for this to be possible, the coefficients g_i and f_j will have to satisfy certain relations. Then, using the description of the Igusa invariants in terms of the coefficients of the curve, we can obtain the equations for the locus of curves in M_2 satisfying these relations.

5.3.3 The stratum of curves with automorphism group C_2^2

Suppose that a general genus two curve admits the automorphism

$$\sigma\colon \begin{cases} x \mapsto -x-1, \\ y \mapsto y. \end{cases} \tag{5.3.1}$$

Then, we can always write such a curve in the form

$$\mathcal{C}\colon \quad y^2 + (g_1x(x+1) + g_0)y = f_3x^3(x+1)^3 + (f_2 - f_1)x^2(x+1)^2 + f_1x(x+1) + f_0$$

for some choice of values of $\{g_0, g_1, f_0, f_1, f_2, f_3\}$.

If the characteristic is two, $W_{\geq C_2^2}$ is given by $\mathbb{V}(h_8, h_9) \subset \mathfrak{X}_0$ where

$$\begin{aligned} h_8 &= J_4^4 + J_4J_6^2 + J_2^2J_4J_8 + J_2^3J_{10}, \\ h_9 &= J_4^3J_6 + J_6^3 + J_2J_4^2J_8 + J_2^2J_4J_{10}. \end{aligned}$$

If the characteristic of the base field is not two, $W_{\geq C_2^2}$ is given by $\mathbb{V}(h_{15}) \subset \mathfrak{X}_0$ where h_{15} is a polynomial of weighted degree 15. This has been described by Shaska and Völklein [SV04].

Parametrisation of the stratum

Regardless of the characteristic of the base field of definition, $W_{\geq C_2^2}$ is an irreducible surface inside M_2. We can also check that the singular subscheme of $W_{\geq C_2^2}$ corresponds to the union of all the strata $W_{\geq H}$ with $H > C_2^2$, which, if the characteristic of the base field is two, is $W_{\geq C_2^5}$ and if the characteristic of the base field is not two, is $W_{\geq D_4} \cup W_{\geq D_6}$.

In characteristic two, we can see that $W_{\geq C_2^2}$ is rational, as we can compute a rational map

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow W_{\geq C_2^2} \subset \mathbb{P}(1,2,3,4,5) \\ [x_0 : x_1 : x_2] &\longmapsto [x_0 : x_0x_1 : x_0x_1^2 : x_0^3x_2 : x_1(x_0x_1^3 + x_1^4 + x_0^3x_2)], \end{aligned}$$

with inverse

$$\begin{aligned} W_{\geq C_2^2} &\longrightarrow \mathbb{P}^2 \\ [J_2 : J_4 : J_6 : J_8 : J_{10}] &\longrightarrow [J_2^4 : J_2^2J_4 : J_8]. \end{aligned}$$

In characteristic zero, it is also known that $W_{\geq C_2^2}$ is rational, for instance, by the work of Kumar [Kum15]. Through similar methods, in every characteristic that is not two, we have computed a morphism

$$\mathbb{P}(1,1,2) \longrightarrow W_{\geq C_2^2} \subset \mathbb{P}(1,2,3,4,5)$$

whose inverse is defined outside the singular locus of $W_{\geq C_2^2}$. This implies, in particular, that $W_{\geq C_2^2}$ is birationally equivalent to $\mathbb{P}(1,1,2)$, and therefore rational.

Let σ as in equation 5.3.1 and $\mathcal{C} \in W_{\geq C_2^2}$. Then, the quotients of $\mathcal{C}$ by either the subgroup $\langle\sigma\rangle \subset \mathrm{Aut}(\mathcal{C})$ or $\langle\iota_h\sigma\rangle \subset \mathrm{Aut}(\mathcal{C})$ are genus one curves, as both quotient maps ramify at two points. There is a natural choice that gives these quotients an elliptic curve structure, which is, for $\mathcal{C}/\langle\sigma\rangle$, to set as the point at infinity the image of the fixed points by $\iota_h\sigma$, and for $\mathcal{C}/\langle\iota_h\sigma\rangle$, to set as the point at infinity the image of the fixed points by σ.

An explicit model for $E_1 = \mathcal{C}/\langle\sigma\rangle$ is

$$E_1\colon \quad y^2 + (g_1x + f_3g_0)y = x^3 + (f_2 - f_1)x^2 + f_1f_3x + f_0f_3^2,$$

where the quotient morphism π_1 is given by

$$\begin{aligned} \pi_1\colon \quad \mathcal{C} &\longrightarrow E_1 \\ (x, y) &\longmapsto (f_3x(x+1), f_3y) \end{aligned}$$

As for $E_2 = \mathcal{C}/\langle\iota_h\sigma\rangle$, if we let

$$\lambda = 64f_0 - 20f_1 + 4f_2 - f_3 + 16g_0^2 - 8g_0g_1 + g_1^2,$$

an explicit model of E_2 is given by

$$E_2\colon \quad y^2 + ((g_1 - 4g_0)x - \lambda g_0)y = x^3 + (48f_0 - 9f_1 + f_2 + 8g_0^2 - 2g_0g_1)x^2 + \lambda(12f_0 - f_1 + g_0^2)x + \lambda^2 f_0.$$

Here,

$$\begin{aligned} \pi_2\colon \quad \mathcal{C} &\longrightarrow E_2 \\ (x, y) &\longmapsto \left(\frac{-\lambda x(x+1)}{(2x+1)^2}, \frac{\lambda(y + (x+1)(g_1x(x+1) + g_0))}{(2x+1)^3}\right) \end{aligned}$$

Let γ be a non-hyperelliptic involution in $\mathrm{Jac}(\mathcal{C})$ and

$$E_\gamma = \{D \in \mathrm{Jac}(\mathcal{C}) : D = (P) + (\gamma(P)) - K_{\mathcal{C}} \text{ with } P \in \mathcal{C}\}.$$

Then, we have two isomorphisms $i_1 : E_1 \to E_\sigma$ and $i_2 : E_2 \to E_{\iota_h\sigma}$. Here is where we can see that our choice of the origin in the E_i is very natural, as the points Q that are fixed by $\iota_h\gamma$ satisfy that $Q = \iota_h\gamma(Q)$ and so,

$$(Q) + (\gamma(Q)) - K_{\mathcal{C}} = (\iota_h\gamma(Q)) + (\gamma(Q)) - K_{\mathcal{C}} = K_{\mathcal{C}} - K_{\mathcal{C}} = 0.$$

We therefore can construct an isogeny

$$\begin{aligned} \psi\colon \quad E_1 \times E_2 &\longrightarrow \mathrm{Jac}(\mathcal{C}) \\ (P_1, P_2) &\longmapsto i_1(P_1) + i_2(P_2) \end{aligned}$$

whose kernel is a subgroup of $\mathrm{Jac}(\mathcal{C})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ [ALT14].

Let $j(E_1)$ and $j(E_2)$ be the j-invariants of E_1 and E_2 respectively. Then, $j(E_1)$ and $j(E_2)$ are both solutions of a quadratic equation of the form

$$h_2 j^2 + h_1 j + h_0 = 0$$

for some $h_i \in k[\mathfrak{X}_0]$. In characteristic two, $j(E_1) = j(E_2)$ only when $J_2 = 0$, which corresponds to the stratum $W_{\geq C_2^5}$. In the rest of the characteristics, the subvariety of $W_{\geq C_2^2}$ for which $j(E_1) = j(E_2)$ forms two curves inside $\mathfrak{X}_0$, one of which corresponds to $W_{\geq D_4}$, and the other corresponding to the curve $Z = \mathbb{V}(h_5, h_6)$ where

$$\begin{aligned} h_5 &= J_2^5 - 56J_2^3J_4 + 912J_2J_4^2 - 3456J_4J_6 + 576J_2J_8 + 17408J_{10}, \\ h_6 &= 3J_2^4J_4 - 150J_2^2J_4^2 + 1871J_4^3 + 27J_6^2 + 73J_2^2J_8 - 1764J_4J_8 + 1904J_2J_{10}. \end{aligned}$$

If the characteristic of the base field is 17, Z is given instead by $\mathbb{V}(h_3, h_{10}) \subseteq \mathfrak{X}_0$ where

$$\begin{aligned} h_3 &= J_2^3 + 2J_2J_4 + 8J_6, \\ h_{10} &= 5J_4^5 + 10J_4^2J_6^2 + J_2^4J_4J_8 + 3J_4^3J_8 + 16J_6^2J_8 + 16J_4J_8^2 + 5J_2J_4^2J_{10} + 9J_4J_6J_{10} + 2J_2J_8J_{10} + 11J_{10}^2. \end{aligned}$$

Connection with the moduli space of pairs of elliptic curves with level 2 structures

In this subsection, let us assume that the characteristic of the base field is not two.

We have seen that for each curve $\mathcal{C} \in W_{\geq C_2^2}$, we can associate a pair of elliptic curves, namely, the quotients of $\mathcal{C}$ by two non-hyperelliptic involutions. A natural question is, is this pair of curves uniquely determined by the choice of $\mathcal{C}$? The answer is no: if $\mathrm{Aut}(\mathcal{C}) = D_4$, then there are multiple choices of non-hyperelliptic involutions, and the resulting quotients can give rise to non-isomorphic pairs of elliptic curves, depending on the choice. An explicit example can be found in the accompanying Magma file in the repository.

We can also ask about the converse: given two elliptic curves, can we construct a genus two curve that is unique up to isomorphism whose quotients are those two curves? The answer to this question is also no, as for each pair of elliptic curves $\{E_1, E_2\}$ there are often multiple ways to glue those elliptic curves along their 2-torsion that give rise to non-isomorphic genus two curves. More specifically, given a pair of elliptic curves, generically, there are six different non-isomorphic genus two curves whose quotients are those two elliptic curves. These six curves correspond to the six possible ways that we can pair the non-trivial 2-torsion points of E_1 and E_2.

However, despite all of this, if we fix extra information, we can still relate the stratum $W_{\geq C_2^2}$ with a coarse moduli space parametrising pairs of elliptic curves. We will now explain how.

Let $\mathcal{M}$ be the moduli space characterising isomorphic pairs $(\mathcal{C}, \sigma)$ where

- The curve $\mathcal{C}$ is in $W_{\geq C_2^2}$.
- σ is a choice of a non-hyperelliptic involution of $\mathcal{C}$.
- Two pairs $(\mathcal{C}, \sigma)$ and $(\mathcal{C}', \sigma')$ are isomorphic if there exists an isomorphism $\alpha : \mathcal{C} \to \mathcal{C}'$ such that $\sigma' = \alpha \circ \sigma \circ \alpha^{-1}$.

Let $X(2)$ be the coarse moduli space of elliptic curves with a **level 2 structure**. The elements of this space correspond to pairs (E, ϕ) where

- E is an elliptic curve.
- ϕ is an isomorphism $\phi : (\mathbb{Z}/2\mathbb{Z})^2 \to E[2](\overline{k})$.
- Two pairs (E, ϕ) and (E', ϕ') are isomorphic if there exists an isomorphism $\alpha : E \to E'$ such that $\phi' = \alpha \circ \phi$.

As a coarse moduli space $X(2)$ is isomorphic to $\mathbb{A}^1 \smallsetminus \{0, 1\}$. This association comes from the fact that every elliptic curve E can be written in Legendre form

$$E_\lambda: \quad y^2 = x(x-1)(x-\lambda)$$

and we can identify $\lambda \leftrightarrow (E_\lambda, \phi_\lambda)$, where the morphism ϕ_λ is set to be the one that sends the two generators of $(\mathbb{Z}/2\mathbb{Z})^2$ to $P_0 = (0,0)$ and $P_1 = (1,0)$ in $E_\lambda[2]$. Note that if we fix E, changing the structure ϕ corresponds to applying one of the following $f_\tau \in \mathrm{PGL}_2(k)$ to λ:

$$\begin{aligned}
f_{id}(\lambda) &= \lambda, & f_{(12)}(\lambda) &= 1-\lambda, & f_{(13)}(\lambda) &= \frac{1}{\lambda}, \\
f_{(23)}(\lambda) &= \frac{\lambda}{\lambda-1}, & f_{(123)}(\lambda) &= \frac{\lambda-1}{\lambda}, & f_{(132)}(\lambda) &= \frac{1}{1-\lambda}.
\end{aligned}$$

Here, $\tau \in S_3$, and this description in terms of f_τ allows us to see S_3 as a subgroup of $\mathrm{PGL}_2(k)$. Now, we define $\overline{\mathcal{M}}_{\text{ell}}$ to be $(X(2) \times X(2))/\operatorname{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$, i.e. the quotient of $X(2) \times X(2)$ under the relation

$$((E_1, \phi_1), (E_2, \phi_2)) \equiv ((E_1', \phi_1'), (E_2', \phi_2'))$$

if and only if $E_1 \cong E_1'$, $E_2 \cong E_2'$, and there is an automorphism $\tau \in \operatorname{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$ such that $\phi_1' = \phi_1 \circ \tau$ and $\phi_2' = \phi_2 \circ \tau$. Note that an automorphism of $(\mathbb{Z}/2\mathbb{Z})^2$ corresponds uniquely to a permutation of the three non-trivial 2-torsion points, and as such, $\operatorname{Aut}((\mathbb{Z}/2\mathbb{Z})^2) \cong S_3$. We will fix this isomorphism to be the one corresponding to the ordering of the non-trivial points of $(\mathbb{Z}/2\mathbb{Z})^2$, $\{(1,0), (0,1), (1,1)\}$, so for the rest of the chapter we can identify any $\tau \in S_3$ as an element of $\operatorname{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$ and vice versa.

In terms of the description of $X(2)$ that we have given, we can describe $\overline{\mathcal{M}}_{\text{ell}}$ as the quotient variety formed by the set of pairs $(\lambda_1, \lambda_2) \in (\mathbb{A}^1 \smallsetminus \{0,1\})^2$ under the relation $(\lambda_1, \lambda_2) \equiv (\lambda_1', \lambda_2')$ if there exists a $\tau \in S_3$ such that $(\lambda_1', \lambda_2') = (f_\tau(\lambda_1), f_\tau(\lambda_2))$.

Let Δ be the subvariety in $\overline{\mathcal{M}}_{\text{ell}}$ formed by the points of the form $((E,\phi),(E,\phi))$ for some $(E,\phi) \in X(2)$ and let $\mathcal{M}_{\text{ell}} = \overline{\mathcal{M}}_{\text{ell}} \setminus \Delta$. We then have the following result:

Theorem 5.3.1. *There is an isomorphism Φ between $\mathcal{M}$ and $\mathcal{M}_{ell}$.*

Proof. For any $\mathcal{C} \in W_{\geq C_2^2}$ and σ a non-hyperelliptic involution, there are ten points that are fixed by some non-trivial element of the group $\langle \iota_h, \sigma \rangle$:

- There are six points fixed by ι_h, namely the Weierstrass points of the curve. These points are not fixed by σ or by $\iota_h\sigma$, but the action of σ permutes them: applying σ to a Weierstrass point yields a different Weierstrass point. Therefore, the six Weierstrass points decompose into three orbits under the action of σ, each consisting of two elements.
- There are two points fixed by σ, which are not fixed by ι_h or $\iota_h\sigma$.
- There are two points fixed by $\iota_h\sigma$, which are not fixed by ι_h or σ.

Recall that given a genus two curve, we can construct morphisms $\pi_1 : \mathcal{C} \to E_1$ and $\pi_2 : \mathcal{C} \to E_2$, where E_1 is the elliptic curve $\mathcal{C}/\langle\sigma\rangle$ with the choice of the point at infinity given by the image by π_1 of the fixed points of $\iota_h\sigma$, and E_2 is the elliptic curve $\mathcal{C}/\langle\iota_h\sigma\rangle$ with the choice of the point at infinity given by the image by π_2 of the fixed points of σ.

The morphism $\Phi : \mathcal{M} \to \mathcal{M}_{\text{ell}}$.

Let $(\mathcal{C},\sigma) \in \mathcal{M}$. As we just mentioned, the action of ι_h on $\mathcal{C}$ fixes the six Weierstrass points, and they form three orbits under the action of σ. Let us fix an ordering of these three orbits:

$$\{\{P_{11},P_{12}\},\{P_{21},P_{22}\},\{P_{31},P_{32}\}\}.$$

Now, let $E_1 = \mathcal{C}/\langle\sigma\rangle$ and $E_2 = \mathcal{C}/\langle\iota_h\sigma\rangle$ as before. The action of ι_h on $\mathcal{C}$ induces an action of order two on E_i, whose fixed points are

$$Q_{i,1} = \pi_i(P_{11}) = \pi_i(P_{12}), \quad Q_{i,2} = \pi_i(P_{21}) = \pi_i(P_{22}), \quad Q_{i,3} = \pi_i(P_{31}) = \pi_i(P_{32}).$$

Therefore, these points correspond to the non-trivial 2-torsion points of E_i. We will then define ϕ_i to be the isomorphism:

$$\begin{aligned}\phi_i\colon \quad (\mathbb{Z}/2\mathbb{Z})^2 &\longrightarrow E_i[2](\overline{k})\\ (1,0) &\longmapsto Q_{i,1}\\ (0,1) &\longmapsto Q_{i,2}\\ (1,1) &\longmapsto Q_{i,3}.\end{aligned}$$

Suppose that we picked another ordering of the orbits of the Weierstrass points. Let $\tilde{Q}_{i,j}$ the points on E_i associated to this order and $\tilde{\phi}_i$ the corresponding level 2 structure.

Then, there exists a $\tau \in S_3$ such that $Q_{i,\tau(j)} = \tilde{Q}_{i,j}$ and, as such, a $\tau \in \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$, such that $\tilde{\phi}_i = \phi_i \circ \tau$.

Furthermore, if we have two isomorphic pairs $(\mathcal{C}, \sigma)$ and $(\mathcal{C}', \sigma')$, there is an isomorphism $\alpha : \mathcal{C} \to \mathcal{C}'$ which induces isomorphisms $\mathcal{C}/\langle\sigma\rangle \cong \mathcal{C}'/\langle\alpha\sigma\alpha^{-1}\rangle = \mathcal{C}'/\langle\sigma'\rangle$ and $\mathcal{C}/\langle\iota_h\sigma\rangle \cong \mathcal{C}'/\langle\alpha\iota_h\sigma\alpha^{-1}\rangle = \mathcal{C}'/\langle\iota_h'\sigma'\rangle$. As α sends the fixed points of $\sigma, \iota_h\sigma$ and ι_h to the fixed points of $\sigma', \iota_h'\sigma'$ and ι_h' respectively, the associated elliptic curves satisfy that $(E_i, \phi_i) \cong (E_i', \phi_i')$.

Therefore, the map

$$\begin{aligned} \Phi\colon \quad \mathcal{M} &\longrightarrow \overline{\mathcal{M}}_{\mathrm{ell}} \cong (X(2) \times X(2))/\mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2) \\ (\mathcal{C}, \sigma) &\longmapsto ((E_1, \phi_1), (E_2, \phi_2)) \end{aligned}$$

is well-defined. The image of Φ lies in $\mathcal{M}_{\mathrm{ell}}$ for the following reason:

Given an elliptic curve E_i, we can always construct a degree two map that ramifies at the four 2-torsion points. As we can map uniquely any two sets of three points in $\mathbb{P}^1$ using a projective transformation, we deduce that for each E_i there exists a unique map $\psi_i : E_i \to \mathbb{P}^1$ satisfying

$$\psi_i(\phi_i((1,0))) = 0, \qquad \psi_i(\phi_i((0,1))) = 1, \qquad \psi_i(\phi_i((1,1))) = \infty.$$

Composing ψ_i with the quotient map $\pi_i : \mathcal{C} \to E_i$ gives us maps $\psi_i \circ \pi_i : \mathcal{C} \to \mathbb{P}^1$ which satisfy that $\psi_1 \circ \pi_1 = \psi_2 \circ \pi_2$.

If $(E_1, \phi_1) \cong (E_2, \phi_2)$, this would imply that there is a $\lambda \in \mathbb{A}^1 \setminus \{0, 1\}$ and two isomorphisms $\alpha_i : E_i \to E_\lambda$ such that $\phi_\lambda = \alpha_i \circ \phi_i$. One can check that if we define ψ' to be the map $E_\lambda \to \mathbb{P}^1$ such that

$$\psi'(\phi_\lambda((1,0))) = 0, \qquad \psi'(\phi_\lambda((0,1))) = 1, \qquad \psi'(\phi_\lambda((1,1))) = \infty,$$

then $\psi' = \psi_1 \circ \alpha_1^{-1} = \psi_2 \circ \alpha_2^{-1}$, and so,

$$\psi' \circ \alpha_1 \circ \pi_1 = \psi_1 \circ \alpha_1^{-1} \circ \alpha_1 \circ \pi_1 = \psi_1 \circ \pi_1 = \psi_2 \circ \pi_2 = \psi_2 \circ \alpha_2^{-1} \circ \alpha_2 \circ \pi_2 = \psi' \circ \alpha_2 \circ \pi_2.$$

Let Q and $\iota_h(Q)$ be the two fixed points of $\iota_h\sigma$ in $\mathcal{C}$. Then, $\alpha_1 \circ \pi_1$ would send both points to the point at infinity of E_λ, which is a ramification point of ψ'. From the description of the fixed points that we gave at the start of the proof, one can check that $\alpha_2 \circ \pi_2$ would send Q and $\iota_h(Q)$ to two different points of E_λ, none of which is a ramification point of ψ'. As ψ' has degree two, we deduce that

$$\psi' \circ \alpha_1 \circ \pi_1(Q) \neq \psi' \circ \alpha_2 \circ \pi_2(Q) \qquad \psi' \circ \alpha_1 \circ \pi_1(\iota_h(Q)) \neq \psi' \circ \alpha_2 \circ \pi_2(\iota_h(Q))$$

and this would lead to a contradiction, as we had shown that $\psi' \circ \alpha_1 \circ \pi_1 = \psi' \circ \alpha_2 \circ \pi_2$.

The morphism $\Psi : \mathcal{M}_{\text{ell}} \to \mathcal{M}$.

Let $((E_1, \phi_1), (E_2, \phi_2)) \in X(2) \times X(2)$ such that $(E_1, \phi_1) \ncong (E_2, \phi_2)$. To construct the inverse of Φ, consider the two morphisms ψ_1 and ψ_2 that we defined before, which, as we saw, depended on the ϕ_i. We define the **fibre product** of (E_1, ϕ_1) and (E_2, ϕ_2) along $\mathbb{P}^1$ to be the scheme $E_1 \times_{\mathbb{P}^1} E_2$, together with morphisms p_1 and p_2 to E_1 and E_2 respectively that satisfies the following universal property: for any scheme W with morphisms φ_1 and φ_2, such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$, there exists a unique morphism $\theta : W \to E_1 \times_{\mathbb{P}^1} E_2$ such that $\varphi_1 = p_1 \circ \theta$ and $\varphi_2 = p_2 \circ \theta$:

$$\begin{tikzcd}
W \arrow[dr, "\exists\theta"] \arrow[drr, bend left, "\varphi_1"] \arrow[ddr, bend right, "\varphi_2"'] & & \\
& E_1 \times_{\mathbb{P}^1} E_2 \arrow[r, "p_1"] \arrow[d, "p_2"] & E_1 \arrow[d, "\psi_1"'] \\
& E_2 \arrow[r, "\psi_2"] & \mathbb{P}^1
\end{tikzcd}$$

We will prove that the normalisation $\mathcal{C}_{E_1 \times_{\mathbb{P}^1} E_2}$ of $E_1 \times_{\mathbb{P}^1} E_2$ is a genus two curve. To do this, we will show that for any E_1 and E_2, there is a genus two curve $\mathcal{C}_{\lambda_1, \lambda_2}$ and morphisms φ_1 and φ_2 making the diagram commutative and use the information that we know about the ramification of the morphisms to show that this curve is isomorphic to $\mathcal{C}_{E_1 \times_{\mathbb{P}^1} E_2}$.

Let $\mu_i \in \mathbb{P}^1$ be the image under ψ_i of the point at infinity of E_i, and let $\lambda_i = \frac{1}{1 - \mu_i}$. As we saw, we can construct an isomorphism $\alpha_i : E_i \to E_{\lambda_i}$ where

$$E_{\lambda_i}\colon \quad y^2 = x(x-1)(x-\lambda_i)$$

such that $(E_i, \phi_i) \cong (E_{\lambda_i}, \phi_{\lambda_i})$ and $\psi_i = \psi_i' \circ \alpha_i$ is defined by

$$\begin{aligned} \psi_i'\colon \quad E_{\lambda_i} &\longrightarrow \mathbb{P}^1 \\ (x, y) &\longmapsto \frac{(1-\lambda_i)x}{x - \lambda_i} \end{aligned}$$

Recall that $(E_1, \phi_1) \ncong (E_2, \phi_2)$ implies that $\lambda_1 \neq \lambda_2$. Now, consider the curve

$$\mathcal{C}_{\lambda_1, \lambda_2}\colon \quad y^2 = (x^2 - 1)\left(x^2 - \frac{\lambda_1}{\lambda_2}\right)\left(x^2 - \frac{\lambda_1(\lambda_2 - 1)}{\lambda_2(\lambda_1 - 1)}\right).$$

By computing its discriminant, we can see that this curve has genus two for any two values $\lambda_1, \lambda_2 \in \mathbb{A}^1 \setminus \{0, 1\}$, such that $\lambda_1 \neq \lambda_2$.

We can construct two maps, ρ_1 and ρ_2

$$\begin{aligned}
\rho_1\colon\quad & \mathcal{C}_{\lambda_1,\lambda_2} \longrightarrow E_{\lambda_1}\\
& (x,y) \longmapsto \left(\nu_1^2\left(x^2 - \frac{\lambda_1(\lambda_2-1)}{\lambda_2(\lambda_1-1)}\right), \nu_1^3 y\right),\\
\rho_2\colon\quad & \mathcal{C}_{\lambda_1,\lambda_2} \longrightarrow E_{\lambda_2}\\
& (x,y) \longmapsto \left(-\nu_2^2\left(\frac{1}{x^2} - \frac{\lambda_2(\lambda_1-1)}{\lambda_1(\lambda_2-1)}\right), \nu_2^3\left(\frac{\lambda_2(\lambda_1-1)^{\frac{1}{2}}y}{\lambda_1(\lambda_2-1)^{\frac{1}{2}}x^3}\right)\right),
\end{aligned}$$

where

$$\nu_1 = \left(\frac{\lambda_2(\lambda_1-1)}{\lambda_1-\lambda_2}\right)^{\frac{1}{2}}, \qquad \nu_2 = \left(\frac{\lambda_1(\lambda_2-1)}{\lambda_1-\lambda_2}\right)^{\frac{1}{2}}.$$

Then, $\psi_1' \circ \rho_1 = \psi_2' \circ \rho_2$, as in both cases, this is the map

$$\begin{aligned}
\psi_i' \circ \rho_i\colon\quad & \mathcal{C}_{\lambda_1,\lambda_2} \longrightarrow \mathbb{P}^1\\
& (x,y) \longmapsto \frac{\lambda_2(1-\lambda_1)x^2 - \lambda_1(1-\lambda_2)}{\lambda_2 x^2 - \lambda_1}.
\end{aligned}$$

As a consequence, we can define $\varphi_1 = \alpha_1^{-1} \circ \rho_1$ and $\varphi_2 = \alpha_2^{-1} \circ \rho_2$, and deduce that there exists a map $\theta : \mathcal{C}_{\lambda_1,\lambda_2} \to E_1 \times_{\mathbb{P}^1} E_2$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{C}_{\lambda_1,\lambda_2} \arrow[rr, "\rho_1"] \arrow[dr, dashed, "\theta"] \arrow[drr, "\varphi_1"] \arrow[dd, "\rho_2"'] \arrow[ddr, "\varphi_2"'] & & E_{\lambda_1} \arrow[dd, bend left, "\psi_1'"] \\
& E_1 \times_{\mathbb{P}^1} E_2 \arrow[r, "p_1"] \arrow[d, "p_2"] & E_1 \arrow[u, "\alpha_1"'] \arrow[d, "\psi_1"'] \\
E_{\lambda_2} \arrow[rr, bend right, "\psi_2'"'] & E_2 \arrow[l, "\alpha_2"'] \arrow[r, "\psi_2"] & \mathbb{P}^1
\end{tikzcd}$$

It is not difficult to see that the preimages of the three points that are simultaneously in the branch loci of ψ_1 and ψ_2 give rise to singularities of $E_1 \times_{\mathbb{P}^1} E_2$, so this is not a smooth curve. As p_1 is a dominant morphism to E_1 of degree two, $E_1 \times_{\mathbb{P}^1} E_2$ has at most two irreducible components. If it had two irreducible components, the ramification locus of p_1 should correspond to the points where the two components meet and, therefore, all of these points should be singular.

However, we can check that p_1 has two ramification points which are smooth, namely, the two preimages of the ramification point of ψ_1 that is not a ramification point of ψ_2 (earlier in the proof I denoted this point as μ_1). Therefore, we deduce that $E_1 \times_{\mathbb{P}^1} E_2$ is irreducible.

Although we will not prove it, one can further check that $E_1 \times_{\mathbb{P}^1} E_2$ has three nodal singularities, and its arithmetic genus is five.

As both maps φ_1 and p_1 have degree two, we deduce that the map θ is generically 1-to-1, and as $\mathcal{C}_{\lambda_1,\lambda_2}$ is smooth, we deduce that $\mathcal{C}_{\lambda_1,\lambda_2}$ is isomorphic to the normalisation of $E_1 \times_{\mathbb{P}^1} E_2$. There is a clear non-hyperelliptic involution in $\mathcal{C}_{\lambda_1,\lambda_2}$ sending $x \mapsto -x$, and this gives rise to an involution $\tilde{\sigma}$ in $\mathcal{C}_{E_1\times_{\mathbb{P}^1}E_2}$. Therefore, we have a morphism

$$\begin{aligned}(X(2)\times X(2))\smallsetminus\Delta &\longrightarrow \mathcal{M}\\ ((E_1,\phi_1),(E_2,\phi_2)) &\longmapsto (\mathcal{C}_{E_1\times_{\mathbb{P}^1}E_2},\tilde{\sigma})\end{aligned}$$

This extends to a morphism $\Psi : \mathcal{M}_{\text{ell}} \to \mathcal{M}$.

To see this, assume that we have $((E_1',\phi_1'),(E_2',\phi_2'))$ such that $E_1 \cong E_1'$, $E_2 \cong E_2'$, and there is an automorphism $\tau \in \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$ with $\tau(\phi_i) = \phi_i'$ for $i = 1,2$. Then, from the description of $X(2)$ that we gave earlier, we see that $\mathcal{C}_{E_1'\times_{\mathbb{P}^1}E_2'} \cong \mathcal{C}_{f_\tau(\lambda_1),f_\tau(\lambda_2)}$. By computing the Igusa invariants with Magma, we can check that

$$J_{2n}(\mathcal{C}_{\lambda_1,\lambda_2}) = J_{2n}(\mathcal{C}_{f_\tau(\lambda_1),f_\tau(\lambda_2)})$$

for all $1 \le n \le 5$ and all $\tau \in S_3$. Therefore, $\mathcal{C}_{\lambda_1,\lambda_2} \cong \mathcal{C}_{f_\tau(\lambda_1),f_\tau(\lambda_2)}$.

Hence, $\Psi\colon \mathcal{M}_{\text{ell}} \to \mathcal{M}_2$ is a well-defined morphism.

Remark 5.3.2. *While it is always true that $\mathcal{C}_{\lambda_1,\lambda_2} \cong \mathcal{C}_{f_\tau(\lambda_1),f_\tau(\lambda_2)}$, it is certainly not true in general that $\mathcal{C}_{\lambda_1,\lambda_2} \cong \mathcal{C}_{\lambda_1,f_\tau(\lambda_2)}$. In fact, one can check that $\mathcal{C} \in W_{\ge D_4}$ if and only if there exists $\lambda \in \mathbb{A}^1 \smallsetminus \{0,1\}$ and an involution $\tau \in S_3$ such that $\mathcal{C} \cong \mathcal{C}_{\lambda,f_\tau(\lambda)}$. Likewise, $\mathcal{C} \in Z$, where Z is as defined in subsection 5.3.3, if and only if there exists λ and an element $\tau \in S_3$ of order three such that $\mathcal{C} \cong \mathcal{C}_{\lambda,f_\tau(\lambda)}$.*

$\underline{\Phi = \Psi^{-1}}$

Finally, to prove that we have an isomorphism $\Phi : \mathcal{M} \to \mathcal{M}_{\text{ell}}$, we need to check that $\Psi \circ \Phi = \mathrm{id}_{\mathcal{M}}$. This is easy to see from the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[drr, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\
& \mathcal{C}/\langle\sigma\rangle \times_{\mathbb{P}^1} \mathcal{C}/\langle\iota_h\sigma\rangle \arrow[r, "p_1"'] \arrow[d, "p_2"] & \mathcal{C}/\langle\sigma\rangle \arrow[d, "\psi_1"'] \\
& \mathcal{C}/\langle\iota_h\sigma\rangle \arrow[r, "\psi_2"] & \mathbb{P}^1
\end{tikzcd}$$

The universal property of the fibre product implies that we can construct a map $\theta : \mathcal{C} \to \mathcal{C}/\langle\sigma\rangle \times_{\mathbb{P}^1} \mathcal{C}/\langle\iota_h\sigma\rangle$. This map θ induces an isomorphism between $\mathcal{C}$ and the normalisation of $\mathcal{C}/\langle\sigma\rangle \times_{\mathbb{P}^1} \mathcal{C}/\langle\iota_h\sigma\rangle$. By carefully tracking the involution σ through the commutative diagrams, one can check that $\Psi \circ \Phi = \mathrm{id}_{\mathcal{M}}$. □

For every $\mathcal{C} \in W_{\geq C_2^2}$ and every non-hyperelliptic involution σ, we have that both $(\mathcal{C}, \sigma)$ and $(\mathcal{C}, \iota_h\sigma)$ are in $\mathcal{M}$, so there is an action of C_2 on $\mathcal{M}$ exchanging these two points of the moduli space. Let $\mathcal{M}/C_2$ denote the quotient. Similarly, we have an action of C_2 on $\mathcal{M}_{\mathrm{ell}}$ that swaps the two copies of $X(2)$. Let $\mathcal{M}_{\mathrm{ell}}/C_2$ be the quotient of $\mathcal{M}_{\mathrm{ell}}$ by that action. We can easily check that

$$\Phi((C,\sigma)) = ((E_1,\phi_1),(E_2,\phi_2)) \quad\iff\quad \Phi((C,\iota_h\sigma)) = ((E_2,\phi_2),(E_1,\phi_1))$$

and therefore C_2 acts on $\mathcal{M}$ and $\mathcal{M}_{\mathrm{ell}}$ in a compatible way. As a consequence, we deduce the following:

Corollary 5.3.3. *As coarse moduli spaces, $\mathcal{M}/C_2$ is isomorphic to $\mathcal{M}_{ell}/C_2$.*

We can define a forgetful morphism $\pi : \mathcal{M}/C_2 \to W_{\geq C_2^2}$ that sends the equivalence class of a pair $(\mathcal{C}, \sigma)$ to $\mathcal{C}$. By checking the conjugacy class of every element of order two in every possible automorphism group of $\mathcal{C}$, we can study the fibres of this morphism π:

1. If $\mathcal{C} \in W_{\geq C_2^2} \smallsetminus W_{\geq D_4}$, we can check that there are only two classes of pairs of points in $\mathcal{M}$ whose first factor is $\mathcal{C}$, corresponding to $(\mathcal{C}, \sigma)$ and $(\mathcal{C}, \iota_h\sigma)$ for some choice of σ. These choices are identified when we take the quotient, and therefore $\pi^{-1}(\mathcal{C})$ is only one point.
2. If $\mathcal{C} \in W_{\geq D_4} \smallsetminus W_{\geq \mathrm{GL}_2(\mathbb{F}_3)}$, there are again two classes of pairs of points in $\mathcal{M}$ whose first factor is $\mathcal{C}$. If we let σ be a non-hyperelliptic involution and τ_4 an automorphism of order four of $\mathcal{C}$, the two pairs are $(\mathcal{C}, \sigma)$ and $(\mathcal{C}, \tau_4\sigma)$. In both cases, $(\mathcal{C}, \sigma) \cong (\mathcal{C}, \iota_h\sigma)$ and $(\mathcal{C}, \tau_4\sigma) \cong (\mathcal{C}, \iota_h\tau_4\sigma)$, so they are different points in $\mathcal{M}/C_2$. Therefore, $\pi^{-1}(\mathcal{C})$ are two points.
3. If $\mathrm{Aut}(\mathcal{C}) = \mathrm{GL}_2(\mathbb{F}_3)$, up to conjugation, $\mathcal{C}$ has a unique non-hyperelliptic involution, and therefore $\pi^{-1}(\mathcal{C})$ is only one point.

From this analysis, we deduce that the forgetful morphism $\pi : \mathcal{M}/C_2 \to W_{\geq C_2^2}$ is injective outside of the points where the image of this map lies inside $W_{\geq D_4} \smallsetminus W_{\geq \mathrm{GL}_2(\mathbb{F}_3)}$. We will use this result in section 5.5.

5.3.4 The stratum of curves with automorphism group D_4

Suppose a general genus two curve admits the automorphism of order four

$$\tau_4 \colon \begin{cases} x \mapsto -x, \\ y \mapsto iy. \end{cases}$$

Then, the characteristic of the base field cannot be two and we can always write such a curve in the form

$$y^2 = f_5x^5 + f_3x^3 + f_1x$$

for some value of $\{f_1, f_3, f_5\}$.

If the characteristic is not five, $W_{\geq D_4}$ is given by $\mathbb{V}(h_5, h_6)$ where

$$\begin{aligned} h_5 &= J_2^5 - 64J_2^3J_4 + 1216J_2J_4^2 - 5760J_4J_6 + 768J_2J_8 - 128000J_{10} \\ h_6 &= J_2^4J_4 - 74J_2^2J_4^2 + 1456J_4^3 - 4320J_6^2 - 40J_2^2J_8 + 1728J_4J_8 - 3200J_2J_{10}, \end{aligned}$$

whereas if the characteristic is five, then $W_{\geq D_4}$ is given by $\mathbb{V}(h_4, h_8, h_9)$ with

$$\begin{aligned} h_4 &= J_2^4 + J_2^2J_4 + J_4^2 + 3J_8, \\ h_8 &= 2J_2^2J_4^3 + 2J_4^4 + 3J_4J_6^2 + 3J_2^2J_4J_8 + J_4^2J_8 + 2J_8^2 + J_2^3J_{10}, \\ h_9 &= J_6^3 + 2J_2J_4^2J_8 + 3J_2J_8^2 + J_2^2J_4J_{10}. \end{aligned}$$

This curve has a singular point in every characteristic. This point is in $\mathfrak{X} \setminus \mathfrak{X}_0$ if the characteristic is three, corresponds to the curve with automorphism group $\mathrm{SL}_2(\mathbb{F}_5)$ if the characteristic is five, and, otherwise, corresponds to the curve with automorphism $C_3 \rtimes D_4$.

Parametrisation of the stratum

Regardless of the characteristic of the base field, the stratum $W_{\geq D_4}$ is a rational curve, as we can find a birational morphism $\mathbb{P}^1 \to W_{\geq D_4}$ whose inverse is given by the map

$$\begin{aligned} W_{\geq D_4} &\longrightarrow \mathbb{P}^1 \\ [J_2 : J_4 : J_6 : J_8 : J_{10}] &\longmapsto [J_2(11J_2^2 - 480J_4) : 5J_2^3 - 224J_2J_4 - 720J_6] \end{aligned}$$

which is defined outside of the singular point of $W_{\geq D_4}$.

Unlike in the general case, all the points $W_{\geq D_4}$ over any field k correspond to genus two curves defined over k. As a matter of fact, from the previous map that we described, we can see that if $P = [J_2 : J_4 : J_6 : J_8 : J_{10}]$ is a smooth point of $W_{\geq D_4}$, a model for the curve corresponding to this point is

$$y^2 = x^5 + x^3 + \left(\frac{7J_2^3 - 288J_2J_4 + 2160J_6}{4(3J_2^3 - 160J_2J_4 - 3600J_6)}\right)x.$$

This expression generalises the one found by Cardona and Quer, in the sense that it is valid in every characteristic that is not two [CQ07, Proposition 2.1].

5.3.5 The stratum of curves with automorphism group D_6

Suppose a general genus two curve admits the automorphism of order three

$$\tau_3\colon \begin{cases} x \mapsto \frac{1}{1-x}, \\ y \mapsto \frac{y}{(x-1)^3}. \end{cases}$$

Then, we can always write such a curve in the form $y^2 + g(x)y = f(x)$ with

$$g(x) = g_0x^3 - (g_1 + 3g_0)x^2 + g_1x + g_0,$$
$$f(x) = f_0x^6 - (f_1 + 6f_0)x^5 + (f_2 + 5f_1 + 15f_0)x^4 - (2f_2 + 5f_1 + 10f_0)x^3 + f_2x^2 + f_1x + f_0,$$

for some choice of values of $\{g_0, g_1, f_0, f_1, f_2\}$.

If the characteristic is three, $W_{\geq D_6}$ is given by $\mathbb{V}(h_2, h_5) \subset \mathfrak{X}_0$ where

$$\begin{aligned} h_2 &= J_4, \\ h_5 &= J_2J_8 - J_{10}. \end{aligned}$$

If the characteristic is five, $W_{\geq D_6}$ is given by $\mathbb{V}(h_4, h_8, h_9) \subset \mathfrak{X}_0$ where

$$\begin{aligned} h_4 &= J_2J_6 + 4J_8, \\ h_8 &= 3J_4J_6^2 + 4J_2^2J_4J_8 + J_8^2 + J_2^3J_{10}, \\ h_9 &= J_6^3 + J_4J_6J_8 + 3J_2J_8^2 + J_2^2J_4J_{10}. \end{aligned}$$

Otherwise, $W_{\geq D_6}$ is given by $\mathbb{V}(h_5, h_6) \subset \mathfrak{X}_0$ where

$$\begin{aligned} h_5 &= 2J_2^2J_6 - 45J_4J_6 - 2J_2J_8 - 375J_{10}, \\ h_6 &= -175J_4^3 + 9J_2^3J_6 - 675J_6^2 - 9J_2^2J_8 - 540J_4J_8. \end{aligned}$$

This curve has a singular point with coordinates $[20 : 30 : -20 : -325 : 64]$ in every characteristic except three. This point is in $\mathfrak{X} \setminus \mathfrak{X}_0$ if the characteristic is two, corresponds to the curve with automorphism group $\mathrm{SL}_2(\mathbb{F}_5)$ if the characteristic is five, and, otherwise, corresponds to the curve with automorphism group $\mathrm{GL}_2(\mathbb{F}_3)$.

Parametrisation of the stratum

As before, regardless of the characteristic of the base field, the stratum $W_{\geq D_6}$ is a rational curve, as we can find a birational morphism $\mathbb{P}^1 \to W_{\geq D_6}$.

The inverse is given by

$$\begin{aligned} W_{\geq D_6} &\longrightarrow \mathbb{P}^1 \\ [J_2 : J_4 : J_6 : J_8 : J_{10}] &\longmapsto [J_2^3 : J_6] \end{aligned}$$

in characteristic three, and

$$\begin{aligned} W_{\geq D_6} &\longrightarrow \mathbb{P}^1 \\ [J_2 : J_4 : J_6 : J_8 : J_{10}] &\longmapsto [J_2(3J_2^2 - 40J_4) : -J_2J_4 - 30J_6] \end{aligned}$$

otherwise. This map is defined outside of the singular point of $W_{\geq D_6}$.

Similarly to the $W_{\geq D_4}$ stratum, all the points $W_{\geq D_6}$ over k correspond to genus two curves defined over k. If the characteristic is not three, we can see that if P is a smooth point of $W_{\geq D_6}$ with coordinates $[J_2 : J_4 : J_6 : J_8 : J_{10}]$, a model for the curve corresponding to this point is

$$y^2 + (x^3 + 1)y = -x^6 - \frac{3J_2^3 - 133J_2J_4 - 2790J_6}{3(3J_2^3 - 160J_2J_4 - 3600J_6)}.$$

This expression generalises the one found by Cardona and Quer, as it is valid in every characteristic that is not three [CQ07, Proposition 2.1].

Extending our base field to include a third root of unity ω, we can find a model for this curve as in the start of subsection 5.3.5 where the values for $\{f_0, f_1, f_2, g_0, g_1\}$ are given by

$$\begin{aligned} f_0 &= \frac{12J_2^3 - 613J_2J_4 - 13590J_6}{81(3J_2^3 - 160J_2J_4 - 3600J_6)}, \\ f_1 &= \frac{2\omega((3+9\omega)J_2^3 - (133+480\omega)J_2J_4 - (2790+10800\omega)J_6)}{27(3J_2^3 - 160J_2J_4 - 3600J_6)}, \\ f_2 &= \frac{5\omega(J_2^3 - (480+133\omega)J_2J_4 + (10800+2790\omega)J_6)}{27(3J_2^3 - 160J_2J_4 - 3600J_6)}, \\ g_0 &= 0, \\ g_1 &= 1. \end{aligned}$$

This model of the curve does not reduce well modulo three, but the Igusa invariants define a point in M_2 that has good reduction at $p = 3$. The reduction of that point modulo three corresponds to the following genus two curve:

$$y^2 + (-x^2 + x)y = -\frac{J_6^{\frac{1}{3}}}{J_2}(x+1)^6.$$

5.3.6 The stratum of curves with automorphism group C_2^5

Suppose that the automorphism group of a genus two curve in characteristic two contains C_2^5. Then, by the work of Igusa, we can see that this curve must be of the form

$$y^2 + g_0 y = f_5 x^5 + f_3 x^3$$

for some value of $\{g_0, f_3, f_5\}$.

The stratum $W_{\geq C_2^5}$ is given by $\mathbb{V}(J_2, J_4, J_6) \subset \mathfrak{X}_0$, so it is parametrised to be the points of the form $[0 : 0 : 0 : J_8 : J_{10}] \in M_2$. If we are working over a perfect field k, the curve corresponding to the point $[0 : 0 : 0 : J_8 : J_{10}]$ is also defined over k and corresponds to setting the values of $\{g_0, f_3, f_5\}$ to

$$g_0 = J_{10}^{\frac{1}{8}}, \qquad f_3 = J_8^{\frac{1}{8}}, \qquad f_5 = 1,$$

in the previous model.

5.4 The Ekedahl-Oort stratification

Let $\mathrm{Jac}(\mathcal{C})$ be the Jacobian variety of a genus two curve $\mathcal{C}$ defined over a field k, which we assume to be perfect. We saw in subsection 1.3.3 the definition of the p-rank. Alternatively, we can define the p-rank in terms of the group scheme μ_p as

$$f(\mathrm{Jac}(\mathcal{C})) = \dim_{\mathbb{F}_p}(\mathrm{Hom}(\mu_p, \mathrm{Jac}(\mathcal{C}))).$$

Analogously to the p-rank, the ***a*-rank** of $\mathrm{Jac}(\mathcal{C})$ is defined to be

$$a(\mathrm{Jac}(\mathcal{C})) = \dim_k(\mathrm{Hom}(\alpha_p, \mathrm{Jac}(\mathcal{C}))).$$

Note that $0 \leq a \leq 2$ and $0 \leq a + f \leq 2$.

In positive characteristic, it is well-known that the moduli space of principally polarised varieties can always be stratified by the p-rank of the abelian variety. This stratification can be refined in many ways, such as intersecting these strata with the ones corresponding to abelian varieties with a fixed a-rank, or by considering the stratification in terms of the Newton polygon of the variety.

However, the information provided by all those strata is determined by an even finer way of stratifying the moduli space in terms of an invariant known as the **Ekedahl-Oort type**.

The idea behind this invariant is that the p-torsion of an abelian variety is a symmetric truncated Barsotti-Tate group scheme of level one (BT_1), and the Ekedahl-Oort type classifies its isomorphism type as a BT_1 group scheme. This information is encoded in combinatorial information corresponding to the possible filtrations of the Dieudonné module associated to the p-torsion.

The theory behind these strata is incredibly rich, and gives rise to many fine strata for abelian varieties of high dimension [vdG99].

However, in the case of abelian surfaces, we will not need to get into the details, as there are only four Ekedahl-Oort strata, and they all correspond to the intersection of strata with p-rank f and a-rank a, as shown by Pries [Pri08]:

Type	f	a	dim
$(\mathbb{Z}/p\mathbb{Z})^2 \oplus \mu_p^2$	2	0	3
$\mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus I_{1,1}$	1	1	2
$I_{2,1}$	0	1	1
$(I_{1,1})^2$	0	2	0

Table 5.2: Types of Ekedahl-Oort strata for abelian surfaces.

Here, $I_{r,1}$ is defined to be the unique symmetric BT_1 group scheme of rank p^{2r} with p-rank 0 and a-number 1.

5.4.1 The Hasse-Witt matrix of a genus two curve

We will now see how to compute the p-rank and a-rank of the Jacobian of a genus two curve $\mathcal{C}$. Assume k is a perfect field of characteristic $p > 0$, let $\sigma : k \to k$ be the Frobenius automorphism and τ be its inverse. Let $H^0(\mathcal{C}, \Omega^1_{\mathcal{C}})$ be the space of regular 1-forms on $\mathcal{C}$, which is a two-dimensional k-vector space that admits the basis $B = \{\frac{dx}{y}, \frac{xdx}{y}\}$. The **Cartier operator** is the semi-linear map $\mathcal{C} : H^0(\mathcal{C}, \Omega^1_{\mathcal{C}}) \to H^0(\mathcal{C}, \Omega^1_{\mathcal{C}})$ satisfying the following properties:

1. $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$.
2. $\mathcal{C}(f^p\omega) = f\mathcal{C}(\omega)$.
3. $\mathcal{C}(f^{n-1}df) = \begin{cases} df & \text{if } n = p, \\ 0 & \text{otherwise.} \end{cases}$

We can check that the Cartier operator is a τ-linear operator, whose associated matrix in the basis B is what we know as the **Cartier-Manin** matrix M_{CM}. Here, we work under the convention that left multiplication by M_{CM} calculates the effect of $\mathcal{C}$ in the basis B [AH19].

On the other hand, by Serre duality, the choice of B fixes a basis B' for the dual space $H^1(\mathcal{C}, \mathcal{O}_\mathcal{C})$. The action of Frobenius on $H^1(\mathcal{C}, \mathcal{O}_\mathcal{C})$ gives then rise to a σ-linear operator, whose associated matrix in the basis B' is known as the **Hasse-Witt matrix** M_{HW}. The way these two matrices are related is

$$M_{HW} = (M_{CM}^{\sigma})^{\top}, \qquad M_{CM} = (M_{HW}^{\tau})^{\top}.$$

In characteristic $p \geq 3$, there is an easy way to compute these two matrices. If we complete the square to make $\mathcal{C}$ to be of the form $y^2 = f(x)$, for some $f(x)$ of degree either five or six, and if we define the c_i to be the constants satisfying the relation

$$\sum_{i=0} c_i x^i = f(x)^{\frac{p-1}{2}},$$

we can check that the Hasse-Witt matrix M_{HW} is given by

$$M_{HW} = \begin{pmatrix} c_{p-1} & c_{2p-1} \\ c_{p-2} & c_{2p-2} \end{pmatrix}.$$

The p-rank and a-rank of the Jacobian of a genus two curve can be easily computed from the Hasse-Witt matrix by the following result:

Proposition 5.4.1. [IKO86, Lemma 1.1] *The Ekedahl-Oort type of* $\mathrm{Jac}(\mathcal{C})$ *is completely determined by its Hasse-Witt matrix:*

- $(f, a) = (2, 0)$ *if and only if* M_{HW} *has rank two.*
- $(f, a) = (1, 1)$ *if and only if* M_{HW} *has rank one and* $M_{HW} M_{HW}^{\sigma} \neq 0$.
- $(f, a) = (0, 1)$ *if and only if* $M_{HW} \neq 0$ *and* $M_{HW} M_{HW}^{\sigma} = 0$.
- $(f, a) = (0, 2)$ *if and only if* $M_{HW} = 0$.

For any $0 \leq f, a \leq 2$, the sets

$$\{X \in \mathcal{A}_2(\bar{k}) : f(X) \leq f\}, \qquad \{X \in \mathcal{A}_2(\bar{k}) : a(X) \geq a\},$$

are closed subschemes of $\mathcal{A}_2$ of pure dimension $f + 1$ and $3 - \left\lfloor \frac{a(a+1)}{2} \right\rfloor$ respectively [vdG99, Theorem 14.7]. As the Hasse-Witt matrix of $\mathrm{Jac}(\mathcal{C})$ can be determined from the coefficients of $\mathcal{C}$, we can also deduce that both

$$V_{\leq f} = \{\mathcal{C} \in M_2(\bar{k}) : f(\mathrm{Jac}(\mathcal{C})) \leq f\}, \qquad T_{\geq a} = \{\mathcal{C} \in M_2(\bar{k}) : a(\mathrm{Jac}(\mathcal{C})) \geq a\},$$

form closed subschemes of M_2, as both the p-rank and the a-rank can be expressed as the vanishing of polynomials involving the coefficients of M_{HW}. From the fact that the Torelli map from M_2 to $\mathcal{A}_2$ is injective and dense, we deduce that $\dim(V_{\leq f}) = f + 1$ and $\dim(T_{\geq a}) = 3 - \left\lfloor \frac{a(a+1)}{2} \right\rfloor$.

5.4.2 Computing the Ekedahl-Oort strata

Using proposition 5.4.1, we propose an algorithm to compute the Ekedahl-Oort strata and their intersections with the automorphism strata.

Assuming that the characteristic of the base field is not two, given any genus two curve, we can always find a projective transformation that sends three of the Weierstrass points to $(0,0)$, $(1,0)$ and ∞ and, as such, we can always transform any curve into one of the form

$$\mathcal{C}\colon \quad y^2 = x(x-1)(a_3x^3 + a_2x^2 + a_1x + a_0).$$

Note that if we replace all the a_i by λa_i for some $\lambda \in k^\times$, we obtain a curve that is also isomorphic to $\mathcal{C}$. Therefore, we can consider $\mathcal{C}$ as the point $[a_0 : a_1 : a_2 : a_3]$ inside $\mathbb{P}^3$. The discriminant of $\mathcal{C}$ is a degree ten polynomial in the variables a_i, and outside of the vanishing locus of this polynomial, each point of $\mathbb{P}^3$ corresponds to a point in M_2. Therefore, we can define a rational map

$$\phi\colon \quad \mathbb{P}^3 \longrightarrow \mathfrak{X}_0$$

by sending a point $[a_0 : a_1 : a_2 : a_3]$ to the Igusa invariants corresponding to the curve defined by those coefficients. Let M_{HW} denote the Hasse–Witt matrix associated to $\mathcal{C}$, which has entries in $k[a_0, a_1, a_2, a_3]$. We define the following subvarieties of $\mathbb{P}^3$:

$$\begin{aligned} X_{(1,1)} &= \mathbb{V}(\det(M_{HW})), \\ X_{(0,1)} &= \mathbb{V}(M_{HW}M_{HW}^{\sigma}), \\ X_{(0,2)} &= \mathbb{V}(M_{HW}). \end{aligned}$$

By proposition 5.4.1, it is easy to check that $\phi(X_{(f,a)}) = V_{\le f} \cap T_{\ge a}$.

This is quite an inefficient way to compute curves in the supersingular strata, and there are better algorithms in the literature for this purpose, such as the one developed by Pieper [Pie22]. However, this algorithm can be used quite successfully to compute the stratum $V_{\le 1} = V_{\le 1} \cap T_{\ge 1}$.

Assuming we are not working in characteristic two, $\mathfrak{X} \cong \mathbb{P}(1,2,3,5)$. Then, using this algorithm, we have proved that $V_{\le 1}$ can be described inside $\mathbb{P}(1,2,3,5)$ as a reduced and irreducible hypersurface of degree $\frac{1}{2}(p-1)$, at least for all primes $p \le 71$.

In the Magma file, we have also managed to compute the singularities of these surfaces and the intersections with the stratum $W_{\ge C_2^2}$. These computations have been very useful for gathering evidence for the statements that are later proved in the following section.

Another consequence of these computations is the following result:

Proposition 5.4.2. *The stratum $V_{\leq 1}$ is rational if $p \leq 19$.*

Proof. Given a well-formed quasismooth hypersurface X_d of weighted degree d inside a weighted projective space whose highest weight is w, Esser proved that X_d is rational whenever $d < 2w$ [Ess24, Proposition 3.1]. In our case, given that $d = \frac{1}{2}(p-1)$ and $w = 5$, we obtain that $V_{\leq 1}$ is rational when $p < 21$. □

The fact that the degree of $V_{\leq 1}$ grows linearly with p is a sign that there may be a prime p_0 such that $V_{\leq 1}$ is not rational for every $p \geq p_0$. However, it is also true that $V_{\leq 1}$ is related to the determinantal variety of a 2×2 matrix, which is a rational surface.

Therefore, we would be interested to know:

Question 5.4.3. *Is $V_{\leq 1}$ rational in characteristic p for all prime numbers?*

5.5 The intersections of the strata

In the previous sections we have described the automorphism and the Ekedahl-Oort strata. Now we will proceed to describe the intersections of these. Consider

$$\begin{aligned} W_{=G} &= \{\mathcal{C} \in M_2(\bar{k}) : \operatorname{Aut}(\mathcal{C}) = G\}, \\ V_{=(f,a)} &= \{\mathcal{C} \in M_2(\bar{k}) : f(\operatorname{Jac}(\mathcal{C})) = f \text{ and } a(\operatorname{Jac}(\mathcal{C})) = a\}. \end{aligned}$$

We can see that these are all strata of M_2 because we can express them as

$$\begin{aligned} W_{=G} &= W_{\geq G} \smallsetminus \bigcup_{H > G} W_{\geq H}, \\ V_{=(f,a)} &= (V_{\leq f} \cap T_{\geq a}) \smallsetminus ((V_{\leq f-1} \cap T_{\geq a}) \cup (V_{\leq f} \cap T_{\geq a+1})), \end{aligned}$$

where $V_{\leq -1} = T_{\geq 3} = \varnothing$. Furthermore, $\overline{W_{=G}} = W_{\geq G}$.

5.5.1 Dimensions of the intersections

We will now identify the empty strata $V_{=(f,a)} \cap W_{=G}$, and compute the dimension of those which are non-empty. In all of the cases, the strata are equidimensional.

Theorem 5.5.1. *Let G be the automorphism group of a genus two curve over an algebraically closed field of characteristic p, and let f, a be two integers between zero and two. Then, the dimensions of the strata $V_{=(f,a)} \cap W_{=G}$, i.e. the loci in M_2 of genus two curves with p-rank f, a-rank a and automorphism group G, are collected in the following tables for each possible value of p.*

When $p = 2$,

		C_2	C_2^2	D_6	C_2^5	$C_2 \wr C_5$
2	0	3	2	1	−1	−1
1	1	2	−1	−1	−1	−1
0	1	−1	−1	−1	1	0
0	2	−1	−1	−1	−1	−1

Table 5.3: Dimensions of $V_{=(f,a)} \cap W_{=G}$ when $p = 2$.

When $p = 3$,

		C_2	C_2^2	D_4	D_6	$\mathrm{GL}_2(\mathbb{F}_3)$	C_{10}
2	0	3	2	1	1	0	−1
1	1	2	1	−1	−1	−1	−1
0	1	1	−1	−1	−1	−1	0
0	2	−1	−1	−1	−1	−1	−1

Table 5.4: Dimensions of $V_{=(f,a)} \cap W_{=G}$ when $p = 3$.

When $p = 5$,

		C_2	C_2^2	D_4	D_6	$\mathrm{SL}_2(\mathbb{F}_5)$
2	0	3	2	1	1	−1
1	1	2	1	−1	−1	−1
0	1	1	−1	−1	−1	−1
0	2	−1	−1	−1	−1	0

Table 5.5: Dimensions of $V_{=(f,a)} \cap W_{=G}$ when $p = 5$.

When $p \geq 7$,

		C_2	C_2^2	D_4	D_6
2	0	3	2	1	1
1	1	2	1	−1	−1
0	1	1	−1	−1	−1
0	2	0 iff $p \geq 29$	0 iff $p \geq 17$	0 iff $p \geq 13$	0 iff $p \geq 11$

Table 5.6: Dimensions of $V_{=(f,a)} \cap W_{=G}$ when $p \geq 7$ (I).

		$C_3 \rtimes D_4$	$\mathrm{GL}_2(\mathbb{F}_3)$	C_{10}
2	0	0 iff $p \equiv 1 \pmod 6$	0 iff $p \equiv 1, 3 \pmod 8$	0 iff $p \equiv 1 \pmod 5$
1	1	-1	-1	-1
0	1	-1	-1	0 iff $p \equiv 2, 3 \pmod 5$
0	2	0 iff $p \equiv 5 \pmod 6$	0 iff $p \equiv 5, 7 \pmod 8$	0 iff $p \equiv 4 \pmod 5$

Table 5.7: Dimensions of $V_{=(f,a)} \cap W_{=G}$ when $p \geq 7$ (II).

Proof. The values for $\dim(V_{=(2,0)} \cap W_{=G})$ where $G \in \{C_2, C_2^2, D_4, D_6\}$ follow from the description of the dimensions of the automorphism strata in section 5.3.

Likewise, the values for $\dim(V_{=(f,a)} \cap W_{=C_2})$ where $(f, a) \in \{(2,0), (1,1), (0,1)\}$ follow from the description of the dimensions of the Ekedahl-Oort strata given in section 5.4.

The p-ranks of the zero-dimensional strata of subsection 5.3.1 were computed by Ibukiyama, Katsura and Oort [IKO86].

Finally, the second and third rows of these tables boil down to the following result:

Proposition 5.5.2. [IKO86, Propositions 1.3 and 1.10] *Let $\mathcal{C}$ be a genus two curve with $\mathrm{Aut}(\mathcal{C}) > C_2^2$. Then, the p-rank of $\mathrm{Jac}(\mathcal{C})$ is either zero or two. Furthermore, if $\mathrm{Jac}(\mathcal{C})$ is supersingular and $\mathrm{Aut}(\mathcal{C}) \geq C_2^2$, then $\mathrm{Jac}(\mathcal{C})$ is superspecial.*

The proof of this relies on the fact that, as we saw, if $\mathrm{Aut}(\mathcal{C}) \geq C_2^2$, then $\mathrm{Jac}(\mathcal{C})$ is isogenous to the product of two elliptic curves. Moreover, if $\mathrm{Aut}(\mathcal{C}) > C_2^2$ one can easily see that these two elliptic curves must be isogenous, and therefore they must have the same p-rank, which implies that $f(\mathrm{Jac}(\mathcal{C})) \neq 1$. As a consequence of this statement, we deduce that $V_{=1,1} \cap W_{=G} = \varnothing$ unless $G = C_2$ or C_2^2, and $V_{=0,1} \cap W_{=G} = \varnothing$ unless $G = C_2$.

Finally, we know that $\dim(V_{=1,1} \cap W_{=C_2^2}) = 1$, because $\dim(V_{\leq 1}) = \dim(W_{\geq C_2^2}) = 2$ and these two strata intersect transversely, as will become apparent in the proof of theorem 5.5.5. □

From this table, we can draw conclusions that may be difficult to prove through other methods. For instance,

Corollary 5.5.3. *There are no Jacobians of genus two curves with p-rank one and an automorphism of order greater than two.*

5.5.2 Irreducible components of the intersections

A natural related question that may arise after theorem 5.5.1 is the following:

Question 5.5.4. *Are the intersections of these strata irreducible? If not, how many different irreducible components does the Zariski closure have?*

The answer is given by the following theorem:

Theorem 5.5.5. *If $p \leq 5$, the Zariski closure of the non-empty strata $V_{=(f,a)} \cap W_{=G}$ are all irreducible.*

If $p \geq 7$ the number of irreducible components of $\overline{V_{=(f,a)} \cap W_{=G}}$ are the following:

		C_2	C_2^2	D_4	D_6
2	0	1	1	1	1
1	1	1	$n_{(1,1),C_2^2}$	0	0
0	1	$n_{(0,1),C_2}$	0	0	0
0	2	$n_{(0,2),C_2}$	$n_{(0,2),C_2^2}$	$n_{(0,2),D_4}$	$n_{(0,2),D_6}$

Table 5.8: Number of irreducible components of $\overline{V_{=(f,a)} \cap W_{=G}}$ when $p \geq 7$ (I).

		$C_3 \rtimes D_4$	$\mathrm{GL}_2(\mathbb{F}_3)$	C_{10}
2	0	1 iff $p \equiv 1 \pmod 6$	1 iff $p \equiv 1, 3 \pmod 8$	1 iff $p \equiv 1 \pmod 5$
1	1	0	0	0
0	1	0	0	1 iff $p \equiv 2, 3 \pmod 5$
0	2	1 iff $p \equiv 5 \pmod 6$	1 iff $p \equiv 5, 7 \pmod 8$	1 iff $p \equiv 4 \pmod 5$

Table 5.9: Number of irreducible components of $\overline{V_{=(f,a)} \cap W_{=G}}$ when $p \geq 7$ (II).

The values of $n_{(f,a),G}$ in the table 5.8 are

$$n_{(1,1),C_2^2} = \frac{p-1}{12} + \frac{1-\left(\frac{-1}{p}\right)}{4} + \frac{1-\left(\frac{-3}{p}\right)}{3},$$

$$n_{(0,1),C_2} = \frac{p^2-1}{2880} + \frac{(p+1)\left(1-\left(\frac{-1}{p}\right)\right)}{64} + \frac{5(p-1)\left(1+\left(\frac{-1}{p}\right)\right)}{192} + \frac{(p+1)\left(1-\left(\frac{-3}{p}\right)\right)}{72} + \frac{(p-1)\left(1+\left(\frac{-3}{p}\right)\right)}{36}$$
$$+ \left(\frac{2}{5} \text{ iff } p \equiv 2, 3 \pmod 5\right) + \left(\frac{1}{4} \text{ iff } p \equiv 3, 5 \pmod 8\right) + \left(\frac{1}{6} \text{ iff } p \equiv 5 \pmod{12}\right),$$

$$n_{(0,2),C_2} = \frac{(p-1)\,(p^2-35p+346)}{2880} - \frac{1-\left(\frac{-1}{p}\right)}{32} - \frac{1-\left(\frac{-2}{p}\right)}{8} - \frac{1-\left(\frac{-3}{p}\right)}{9} - \left(\frac{1}{5} \text{ iff } p \equiv 4 \pmod 5\right),$$

$$n_{(0,2),C_2^2} = \frac{(p-1)(p-17)}{48} + \frac{1-\left(\frac{-1}{p}\right)}{8} + \frac{1-\left(\frac{-2}{p}\right)}{2} + \frac{1-\left(\frac{-3}{p}\right)}{2},$$

$$n_{(0,2),D_4} = \frac{p-1}{8} - \frac{1-\left(\frac{-1}{p}\right)}{8} - \frac{1-\left(\frac{-2}{p}\right)}{4} - \frac{1-\left(\frac{-3}{p}\right)}{2},$$

$$n_{(0,2),D_6} = \frac{p-1}{6} - \frac{1-\left(\frac{-2}{p}\right)}{2} - \frac{1-\left(\frac{-3}{p}\right)}{3},$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Proof. We have seen in section 5.3 that if $G = C_2, C_2^2, D_4$ or D_6, $\overline{V_{=(2,0)} \cap W_{=G}} = W_{\geq G}$ are all irreducible.

We know that $\overline{V_{=(1,1)} \cap W_{=C_2}} = T_{\geq 1}$ is irreducible as a consequence of the work of van der Geer, who proved that $T_{\geq a} \subset \mathcal{A}_g$ is irreducible whenever $a < g$ [vdG99, Theorem 2.11].

The fact that the supersingular locus of $\mathcal{A}_2$ is not irreducible when $p \geq 13$ is due to Katsura and Oort, who proved that the number of irreducible components $n_{(0,1),C_2}$ is $H_2(1,p)$, the class number of the non-principal genus [KO87]. The closed formula for $H_2(1,p)$ has been computed by Hashimoto and Ibukiyama [HI80].

The values of all the zero-dimensional components (the last row and the last three columns) are again work of Ibukiyama, Katsura and Oort [IKO86, Propositions 1.11-1.13 and Theorem 3.3].

Our main contribution to this table is computing the value of $n_{(1,1),C_2^2}$. □

Proposition 5.5.6. *The stratum $W_{\geq C_2^2} \cap V_{\leq 1}$ is the union of n rational curves, where n is the number of supersingular elliptic curves in characteristic p, which is $n = 1$ if $p < 7$ and*

$$n = \frac{p-1}{12} + \frac{1 - \left(\frac{-1}{p}\right)}{4} + \frac{1 - \left(\frac{-3}{p}\right)}{3}$$

otherwise.

Proof. When $p = 2$, we can use the results in section 5.4 to show that $V_{\leq 1} = \mathbb{V}(J_2, J_4)$, and by computing the intersection with $W_{\geq C_2^2}$, we get

$$W_{\geq C_2^2} \cap V_{\leq 1} = W_{\geq C_2^5} = V_{\leq 0} = \mathbb{V}(J_2, J_4, J_6),$$

which is irreducible.

If $p \neq 2$, from the description in subsection 5.3.3, we saw that there was a morphism $\pi : \mathcal{M}_{\mathrm{ell}}/C_2 \to W_{\geq C_2^2}$ which was an isomorphism when restricted to the open subset $W_{\geq C_2^2} \setminus W_{D_4}$. We also proved that there was an isomorphism Φ between $\mathcal{M}/C_2$ and $\mathcal{M}_{\mathrm{ell}}/C_2$, which is the quotient of

$$\mathcal{M}_{\mathrm{ell}} = ((X(2) \times X(2)) \setminus \Delta)/S_3$$

by the action of C_2 that swaps around the copies of $X(2)$. Furthermore, we saw in section 5.3.3 that if $\Phi((\mathcal{C}, \sigma)) = \{(E_1, \phi_1), (E_2, \phi_2)\}$, $\mathrm{Jac}(\mathcal{C})$ is isogenous to $E_1 \times E_2$. Therefore, $\mathcal{C} \in W_{\geq C_2^2} \cap V_{\leq 1}$ if and only if $f(E_1 \times E_2) \leq 1$. As $f(E_1 \times E_2) = f(E_1) + f(E_2)$, we deduce that this happens if and only if at least one of E_1 or E_2 is supersingular.

Let E_0 be a supersingular elliptic curve, and ϕ_0 a level 2 structure on E_0. Consider the following subvariety

$$E_0 \times X(2) = \{\{(E_0, \phi_0), (E, \phi)\} \text{ for some } (E, \phi) \in X(2) \text{ with } (E, \phi) \neq (E_0, \phi_0)\} \subset \mathcal{M}_{\text{ell}}/C_2.$$

Note that the subscheme $E_0 \times X(2)$ does not depend on the choice of the level 2 structure ϕ_0. This is because, as we have seen, if ϕ_0 and ϕ_0' are two structures on E_0, there exists a $\tau \in \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$ such that $\phi_0' = \phi_0 \circ \tau$. But then, we have that for any $(E, \phi) \in X(2)$,

$$\{(E_0, \phi_0'), (E, \phi)\} = \{(E_0, \phi_0), (E, \phi \circ \tau^{-1})\}$$

in $\mathcal{M}_{\text{ell}}/C_2$. We know that $E_0 \times X(2)$ is irreducible, as it is the image under the quotient map of a variety isomorphic to $X(2)$, which is irreducible.

Let $C_{E_0} := \Phi^{-1}(E_0 \times X(2)) \subset \mathcal{M}/C_2$. From the fact that Φ is an isomorphism and that any pair of elliptic curves where at least one of them is supersingular lies inside a copy of $E_0 \times X(2)$ for some E_0, we deduce that any pair $(\mathcal{C}, \sigma)$ with $\mathcal{C} \in W_{\geq C_2^2} \cap V_{\leq 1}$ must lie inside a C_{E_0} for some supersingular E_0. From the fact that the morphism $\pi : \mathcal{M}_{\text{ell}}/C_2 \to W_{\geq C_2^2}$ is finite and surjective, we deduce that $\pi(C_{E_0})$ is also an irreducible curve, and that

$$W_{\geq C_2^2} \cap V_{\leq 1} = \bigcup_{E_0 \text{ supersingular}} \pi(C_{E_0}).$$

Hence, the number of irreducible components of $W_{\geq C_2^2} \cap V_{\leq 1}$ is the same as the number of supersingular elliptic curves in characteristic p. This quantity was computed by Igusa [Igu58]. $\square$

6 Generalised Kummer surfaces of Jacobians of genus two curves

6.1 Introduction

In chapter 3, we saw many examples of generalised Kummer surfaces defined as quotients of products of elliptic curves. However, as we saw in the previous chapter, most abelian surfaces arise as Jacobians of genus two curves, so we would also like to study generalised Kummer surfaces in this set-up.

There are significant challenges associated with working with Jacobians of genus two curves $\mathcal{C}$, as we discussed in chapter 4. The first is that embedding Jacobians into projective space requires working with many equations in a high-dimensional ambient space. The second is that, while computing the group law on a product of elliptic curves is straightforward, doing so on $\mathrm{Jac}(\mathcal{C})$ is substantially more difficult. This makes it hard to explicitly describe subgroups $G \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(\mathcal{C}))$ and therefore to compute the quotient of $\mathrm{Jac}(\mathcal{C})/G$.

The first challenge can be overcome by working with an alternative embedding of the Jacobian which, rather than lying in $\mathbb{P}^{15}$, lies in the weighted projective space $\mathbb{P}(1^4, 2^6)$, as we will see in section 6.3.

To address the second challenge, we will only be studying quotients by automorphisms of $\mathrm{Jac}(\mathcal{C})$ that preserve the polarisation. As mentioned in the previous chapter, these are in correspondence with the automorphisms of $\mathcal{C}$, and from section 5.3, we have a classification of all such automorphisms.

Combining these two approaches, we will produce new explicit examples of generalised Kummer surfaces. Unlike in section 3.2, where the quotients were constructed by computing invariants through intricate methods, in this chapter, we will adopt a more systematic approach. We will make use of the fact that $\mathrm{Jac}(\mathcal{C})$ comes equipped with the ample line bundle $\Theta_+ + \Theta_-$ and apply the theory of linearisations developed in subsection 1.5.5.

6.2 Polarised and symplectic automorphisms

Let A be an abelian surface and let G be a finite group acting on A via group homomorphisms, so that $G \hookrightarrow \mathrm{End}(A)^\times$.

Definition 6.2.1. *We say that an automorphism $g \in G$ of a principally polarised abelian surface A **preserves the polarisation** λ_A if $g^*\lambda_A = \lambda_A$.*

In the case where $A = E_1 \times E_2$ with the product polarisation, it is easy to see that g preserves the polarisation if and only if it acts independently on E_1 and E_2 without mixing the factors. For example, the action ι defined in subsection 3.2.1 preserves the polarisation, whereas the actions of τ_3, τ_4, and τ_6 in subsections 3.2.2, 3.2.3, and 3.2.4, respectively, do not.

For Jacobians of genus two curves, the group of automorphisms of $\mathrm{Jac}(\mathcal{C})$ preserving the polarisation is isomorphic to the automorphism group of $\mathcal{C}$. This follows from the injectivity of the Torelli map. More precisely, we have the following general result:

Theorem 6.2.2 ([Mil86, Theorem 12.1]). *Let $\mathcal{C}$ and $\mathcal{C}'$ be smooth curves over an algebraically closed field k, and let $\phi : \mathcal{C} \to \mathrm{Jac}(\mathcal{C})$ and $\phi' : \mathcal{C}' \to \mathrm{Jac}(\mathcal{C}')$ be the embeddings into their respective Jacobians. Let $\beta : (\mathrm{Jac}(\mathcal{C}), \lambda) \to (\mathrm{Jac}(\mathcal{C}'), \lambda')$ be an isomorphism of principally polarised abelian varieties. Then,*

1. *There exists an isomorphism $\alpha: \mathcal{C} \to \mathcal{C}'$ such that*

$$\phi' \circ \alpha = \pm\beta \circ t_P$$

where t_P denotes translation by some $P \in \mathrm{Jac}(\mathcal{C}')(k)$.

2. *Assume that $\mathcal{C}$ has genus ≥ 2. If $\mathcal{C}$ is non-hyperelliptic, then the map α, the sign $\pm$, and the point P are uniquely determined by β, ϕ, and ϕ'. If $\mathcal{C}$ is hyperelliptic, the sign can be chosen arbitrarily; then α and P are uniquely determined.*

Whenever we have an automorphism $\alpha : \mathcal{C} \to \mathcal{C}$, an embedding ϕ of $\mathcal{C}$ into its Jacobian gives us an automorphism β of $(\mathrm{Jac}(\mathcal{C}), \lambda)$. What this theorem tells us is the other direction, which is that given a β that fixes $O \in \mathrm{Jac}(\mathcal{C})$ (so that $t_P = \mathrm{id}$), if our curve is hyperelliptic, there is a unique automorphism α of $\mathcal{C}$ such that $\phi \circ \alpha = \beta$. If our curve is not hyperelliptic, then β is either $\phi \circ \alpha$ or $-\phi \circ \alpha$ for some automorphism α of $\mathcal{C}$.

As we saw in chapter 2, one of the conditions required for the quotient $\mathrm{Jac}(\mathcal{C})/G$ to be a K3 surface is that G acts **symplectically** on $\mathrm{Jac}(\mathcal{C})$, meaning that for all $g \in G$, $g^*\omega_{\mathrm{Jac}(\mathcal{C})} = \omega_{\mathrm{Jac}(\mathcal{C})}$.

Let $\mathcal{C}$ be a genus two curve of the form $y^2 + g(x)y = f(x)$ over an algebraically closed field k. Then a basis of global differentials is given by $\langle \omega_1, \omega_2 \rangle \subset H^0(\mathcal{C}, \Omega^1)$, where

$$\omega_1 = \frac{dx}{2y + g(x)}, \qquad \omega_2 = \frac{x\,dx}{2y + g(x)}.$$

A basis for $H^0(\mathrm{Jac}(\mathcal{C}), \Omega^2)$ is then given by $\omega_1 \wedge \omega_2$.

Given an element g of order n acting on $\mathrm{Jac}(\mathcal{C})$, we have that $g^*\omega_{\mathrm{Jac}(\mathcal{C})}$ must be equal to $\zeta_m\,\omega_{\mathrm{Jac}(\mathcal{C})}$ for some m-th root of unity ζ_m such that $m \mid n$. If we define the character

$$\begin{aligned} \chi_{2,0}\colon \quad \mathrm{Aut}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow k^\times \\ g &\longmapsto \frac{g^*\omega_{\mathrm{Jac}(\mathcal{C})}}{\omega_{\mathrm{Jac}(\mathcal{C})}} \end{aligned}$$

it is clear from the definition that g acts symplectically if and only if $\chi_{2,0}(g) = 1$.

6.2.1 Symplectic automorphisms of Jacobians of genus two curves

In the previous chapter, we described all possible automorphisms of a genus two curve $\mathcal{C}$. Let $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ denote the group of (geometric) symplectic automorphisms of the Jacobian of $\mathcal{C}$ that also preserve the polarisation.

Our goal in this section is to determine the possible values of $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$. To achieve this, we will proceed as follows:

Let $G = \mathrm{Aut}(\mathcal{C})$, which, via the Torelli theorem, embeds into $\mathrm{Aut}(\mathrm{Jac}(\mathcal{C}))$ as the group of automorphisms preserving the polarisation. For each g that is a generator of G, we will compute $\chi_{2,0}(g)$ by studying how g acts on the space of global 2-forms of $\mathrm{Jac}(\mathcal{C})$, which is generated by $\omega_1 \wedge \omega_2$.

Then, the group of symplectic automorphisms is precisely the kernel of this character: $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = \ker(\chi_{2,0})$.

- If $\mathrm{Aut}(\mathcal{C}) = C_2$, its only automorphism is the hyperelliptic involution ι_h, which one can easily check is always symplectic, so $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = C_2$.

- If $\mathrm{Aut}(\mathcal{C}) = C_2^2$, we saw in the previous section that we can always write $\mathcal{C}$ as

$$\mathcal{C}\colon \quad y^2 + (g_1x(x+1) + g_0)y = f_3x^3(x+1)^3 + (f_2 - f_1)x^2(x+1)^2 + f_1x(x+1) + f_0$$

  for some choice of coefficients $g_0, g_1, f_0, f_1, f_2, f_3 \in k$, and $\mathrm{Aut}(\mathcal{C}) = \langle \iota_h, \sigma \rangle$ where σ is the non-hyperelliptic involution:

$$\sigma\colon \begin{cases} x \mapsto -x-1, \\ y \mapsto y. \end{cases}$$

Then,

$$\sigma^*(\omega_1) = \frac{d(-x-1)}{2y + (g_1x(x+1) + g_0)} = -\omega_1, \quad \sigma^*(\omega_2) = \frac{(-x-1)d(-x-1)}{2y + (g_1x(x+1) + g_0)} = \omega_1 + \omega_2,$$

so

$$\sigma^*(\omega_1 \wedge \omega_2) = ((-\omega_1) \wedge (\omega_1 + \omega_2)) = -\omega_1 \wedge \omega_2,$$

and, therefore, $\chi_{2,0}(\sigma) = -1$. As a consequence, $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = \ker(\chi_{2,0})$ is C_2 if the characteristic of the base field is not two, and C_2^2 if the characteristic of the base field is two.

- If $\mathrm{Aut}(\mathcal{C}) = D_4$, we can always write $\mathcal{C}$ as

$$\mathcal{C}\colon \quad y^2 = x^5 + x^3 + tx$$

for some $t \in k \setminus \{0, \frac{1}{4}\}$, and $\mathrm{Aut}(\mathcal{C}) = \langle \tau_2, \tau_4 \mid \tau_2^2, \tau_4^4, (\tau_2\tau_4)^2 \rangle$, where

$$\tau_2\colon \begin{cases} x \mapsto \frac{\sqrt{t}}{x}, \\ y \mapsto \frac{\sqrt[4]{t^3}y}{x^3}, \end{cases} \qquad \tau_4\colon \begin{cases} x \mapsto -x, \\ y \mapsto iy. \end{cases}$$

Then, $\chi_{2,0}(\tau_2) = -1$ and $\chi_{2,0}(\tau_4) = 1$ and, therefore, $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = C_4$.

- If $\mathrm{Aut}(\mathcal{C}) = C_{10}$, then up to isomorphism, in characteristics not two or five, $\mathcal{C}$ is

$$\mathcal{C}\colon \quad y^2 + y = x^5$$

and the generator of its automorphism group is

$$\tau_{10}\colon \begin{cases} x \mapsto \zeta_5 x, \\ y \mapsto -y - 1. \end{cases}$$

We can check that $\chi_{2,0}(\tau_{10}) = \zeta_5^3$, so $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = C_2$.

- If $\mathrm{Aut}(\mathcal{C}) = D_6$, when the characteristic is not three, we can always write $\mathcal{C}$ as

$$\mathcal{C}\colon \quad y^2 + (x^3 + 1)y = -x^6 - t$$

for some $t \in k \setminus \{\frac{1}{4}, \frac{1}{3}\}$, and $\mathrm{Aut}(\mathcal{C}) = \langle \tau_2, \tau_6 \mid \tau_2^2, \tau_6^6, (\tau_2\tau_6)^2 \rangle$, where

$$\tau_2\colon \begin{cases} x \mapsto \frac{\sqrt[3]{4t-1}}{\sqrt[3]{3}x}, \\ y \mapsto -\frac{2\sqrt{12t-3}\,y + (3+\sqrt{12t-3})\,x^3 + 4t - 1 + \sqrt{12t-3}}{6x^3}, \end{cases} \qquad \tau_6\colon \begin{cases} x \mapsto \zeta_3 x, \\ y \mapsto -y - x^3 - 1. \end{cases}$$

Note that if the characteristic is two, there is an alternate expression for τ_2.

When the characteristic is three, we can write $\mathcal{C}$ as

$$\mathcal{C}\colon \quad y^2 + (-x^2 + x)y = t(x+1)^6,$$

for some $t \in k \smallsetminus \{0\}$, and

$$\tau_2\colon \begin{cases} x \mapsto \frac{1}{x}, \\ y \mapsto -\frac{y}{x^3}, \end{cases} \qquad \tau_6\colon \begin{cases} x \mapsto \frac{1}{1-x}, \\ y \mapsto \frac{-y+x^2-x}{(x-1)^3}. \end{cases}$$

Regardless of the characteristic, one can check that $\chi_{2,0}(\tau_2) = -1$ and $\chi_{2,0}(\tau_6) = 1$ so $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = C_6$.

- If $\mathrm{Aut}(\mathcal{C}) = C_3 \rtimes D_4$, then up to isomorphism, in characteristics not two, three or five, $\mathcal{C}$ is

$$\mathcal{C}\colon \quad y^2 = x^6 + 1$$

and $\mathrm{Aut}(\mathcal{C}) = \langle \iota_h, \tau_2, \tau_6 \mid \iota_h^2, \tau_2^2, \tau_6^6, [\iota_h, \tau_2], [\tau_2, \tau_6], \iota_h\tau_6\iota_h\tau_6^{-5}\tau_2 \rangle$ where

$$\iota_h\colon \begin{cases} x \mapsto x, \\ y \mapsto -y, \end{cases} \qquad \tau_2\colon \begin{cases} x \mapsto \frac{1}{x}, \\ y \mapsto \frac{y}{x^3}, \end{cases} \qquad \tau_6\colon \begin{cases} x \mapsto \zeta_6 x, \\ y \mapsto y. \end{cases}$$

We can check that $\chi_{2,0}(\iota_h) = 1$ and $\chi_{2,0}(\tau_2) = \chi_{2,0}(\tau_6) = -1$. Therefore, $\chi_{2,0}$ takes values on $\{\pm 1\}$ and $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ is a normal subgroup of $C_3 \rtimes D_4$ of index 2. There are three possibilities: D_6, $C_2 \times C_6$ and $C_3 \rtimes C_4 \cong Q_{12}$, but based on the fact that $\tau_2\tau_6 \in \mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ has order four, this shows that $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = Q_{12}$.

- If $\mathrm{Aut}(\mathcal{C}) = \mathrm{GL}_2(\mathbb{F}_3)$, then up to isomorphism, in characteristics not two or five, $\mathcal{C}$ is

$$\mathcal{C}\colon \quad y^2 = x^5 - x$$

and $\mathrm{Aut}(\mathcal{C}) = \langle \tau_3, \sigma_3, \tau_2 \rangle$ where

$$\tau_3\colon \begin{cases} x \mapsto \frac{-i(x-1)}{x+1}, \\ y \mapsto \frac{(2i+2)y}{(x+1)^3}, \end{cases} \qquad \sigma_3\colon \begin{cases} x \mapsto \frac{-x-i}{x-i}, \\ y \mapsto \frac{(2i-2)y}{(x-i)^3}, \end{cases} \qquad \tau_2\colon \begin{cases} x \mapsto \frac{x+1}{x-1}, \\ y \mapsto \frac{(2\zeta_8^3-2\zeta_8)y}{(x-1)^3}. \end{cases}$$

and the isomorphism between $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{GL}_2(\mathbb{F}_3)$ can be given by

$$\tau_3 \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_3 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \tau_2 \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can check that $\chi_{2,0}(\tau_3) = \chi_{2,0}(\sigma_3) = 1$ and $\chi_{2,0}(\tau_2) = -1$. Following the same argument as in the previous case, $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ must be a normal subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$ of index 2, and therefore, one can check that $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = \mathrm{SL}_2(\mathbb{F}_3)$.

- If $\mathrm{Aut}(\mathcal{C}) = C_2^5$, as this can only happen in characteristic two and all elements have order two, $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = C_2^5$ as well.

- If $\mathrm{Aut}(\mathcal{C}) = C_2 \wr C_5$, up to isomorphism, $\mathcal{C}$ is the curve in characteristic two

$$\mathcal{C}\colon \quad y^2 + y = x^5$$

Let $a \in \mathbb{F}_{16}$ and let b and $b+1$ be the two roots of the equation $b^2 + b + a^5 = 0$. We then have the following automorphisms:

$$\iota_h\colon \begin{cases} x \mapsto x, \\ y \mapsto -y-1, \end{cases} \qquad \phi_{a,b}\colon \begin{cases} x \mapsto x + a, \\ y \mapsto y + a^8x^2 + a^4x + b, \end{cases} \qquad \tau_5\colon \begin{cases} x \mapsto \zeta_5 x, \\ y \mapsto y. \end{cases}$$

As ι_h and $\phi_{a,b}$ are both involutions, and

$$\iota_h\phi_{a,b} = \phi_{a,b}\iota_h = \phi_{a,b+1}, \qquad \phi_{a,b}\phi_{a',b'} = \phi_{a',b'}\phi_{a,b} = \phi_{a+a',b+b'},$$

we deduce that ι_h and the automorphisms of the form $\phi_{a,b}$ generate a subgroup isomorphic to C_2^5. It is then easy to see that the group C_5 generated by τ_5 acts on C_2^5 by conjugation as $\tau_5\iota_h\tau_5^{-1} = \iota_h$ and $\tau_5\phi_{a,b}\tau_5^{-1} = \phi_{\zeta_5 a,b}$.

Because we are in characteristic two, $\chi_{2,0}(\iota_h) = \chi_{2,0}(\phi_{a,b}) = 1$. As $\chi_{2,0}(\tau_5) = \zeta_5^3$, we deduce that $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = C_2^5$.

- If $\mathrm{Aut}(\mathcal{C}) = \mathrm{SL}_2(\mathbb{F}_5)$, up to isomorphism, $\mathcal{C}$ is the curve in characteristic five

$$\mathcal{C}\colon \quad y^2 = x^5 - x$$

and $\mathrm{Aut}(\mathcal{C}) = \langle \tau_5, \sigma_5 \rangle$ where

$$\tau_5\colon \begin{cases} x \mapsto x+1, \\ y \mapsto y. \end{cases} \qquad \sigma_5\colon \begin{cases} x \mapsto \frac{x-1}{-x+2}, \\ y \mapsto \frac{-y}{(-x+2)^3}. \end{cases}$$

An isomorphism between $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{SL}_2(\mathbb{F}_5)$ can be given by

$$\tau_5 \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_5 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

As we are in characteristic five, $\chi_{2,0}(\tau_5) = \chi_{2,0}(\sigma_5) = 1$ and, therefore, we deduce that $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = \mathrm{SL}_2(\mathbb{F}_5)$.

As a consequence of these computations, we have the following result:

Proposition 6.2.3. *Let* $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ *denote the group of (geometric) automorphisms of the Jacobian of a genus two curve that are symplectic. Then, depending on the characteristic of the base field* k *over which* $\mathcal{C}$ *is defined, we have the following possibilities:*

- *If* $\mathrm{char}(k) = 2$*, then* $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ *is one of the groups* C_2*,* C_2^2*,* D_6*, or* C_2^5*.*
- *If* $\mathrm{char}(k) = 3$*, then* $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ *is one of the groups* C_2*,* C_4*,* C_6*, or* $\mathrm{SL}_2(\mathbb{F}_3)$*.*
- *If* $\mathrm{char}(k) = 5$*, then* $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ *is one of the groups* C_2*,* C_4*,* C_6*, or* $\mathrm{SL}_2(\mathbb{F}_5)$*.*
- *If* $\mathrm{char}(k) \geq 7$ *or* $\mathrm{char}(k) = 0$*, then* $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ *is one of the groups* C_2*,* C_4*,* C_6*,* Q_{12}*, or* $\mathrm{SL}_2(\mathbb{F}_3)$*.*

From this proposition we deduce that, assuming $\mathrm{char}(k) \nmid |G|$, the only possibilities for subgroups $G \leq \mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ are C_2, C_4, C_6, Q_{12}, or $\mathrm{SL}_2(\mathbb{F}_3)$. Therefore, we obtain the following:

Corollary 6.2.4. *Let* $G \leq \mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ *be a subgroup such that the characteristic,* p*, does not divide* $|G|$*. Then* G *is either* C_2*,* C_3*,* C_4*,* C_6*,* Q_8*,* Q_{12} *or* $\mathrm{SL}_2(\mathbb{F}_3)$*.*

We will see that, when $p \nmid |G|$, the action of G on $\mathrm{Jac}(\mathcal{C})$ is always rigid and the singularities of the quotient are rational double points. As for the cases where $p \mid |G|$, computing models for $\mathrm{Kum}_G(\mathrm{Jac}(\mathcal{C}))$ is still work in progress. We will briefly comment on this in section 6.5.

The list of possible groups in corollary 6.2.4 has an interesting feature: except for C_3, all of the groups are of even order, and in all these cases the subgroup $\langle \iota_h \rangle$ is normal in G. This implies that if $G \neq C_3$, the quotient map

$$\mathrm{Jac}(\mathcal{C}) \to \mathrm{Kum}_G(\mathrm{Jac}(\mathcal{C}))$$

factors through the classical Kummer surface $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$, and the induced map

$$\mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) \to \mathrm{Kum}_G(\mathrm{Jac}(\mathcal{C}))$$

is the quotient of $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ by the residual action of the group $H = G/\langle \iota_h \rangle$.

6.3 Computing an embedding for the quotient by the action of C_3

Recall from section 4.2 that there is a divisor $\Theta_+ + \Theta_-$ on $\mathrm{Jac}(\mathcal{C})$ such that the linear systems $|\Theta_+ + \Theta_-|$ and $|2(\Theta_+ + \Theta_-)|$ define embeddings of $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ and $\mathrm{Jac}(\mathcal{C})$, respectively, into projective space. Assume for the moment that the characteristic of the base field is not two. In that case, recall that if we denote by $\{k_1, \ldots, k_4\}$ a basis of $\mathcal{L}(\Theta_+ + \Theta_-)$ and by $\{b_1, \ldots, b_6\}$ a basis for the odd functions in $\mathcal{L}(2(\Theta_+ + \Theta_-))$, then a basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ is given by $\{k_1^2, k_1k_2, \ldots, k_4^2, b_1, \ldots, b_6\}$.

Given a curve $\mathcal{C}$ with an action by C_3, we can compute the induced action on $\mathrm{Jac}(\mathcal{C})$ by studying its action on $\mathcal{L}(2(\Theta_+ + \Theta_-))$. We can then compute the invariant functions under this action to construct the quotient. However, this is not a very practical approach, as it results in a model of the quotient embedded in a high-dimensional ambient space.

To bypass this difficulty, we instead compute an alternative embedding of the Jacobian into weighted projective space. This embedding arises from the fact that $\Theta_+ + \Theta_-$ is ample, and therefore

$$\mathrm{Jac}(\mathcal{C}) = \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathcal{L}(n(\Theta_+ + \Theta_-))\right).$$

As Mumford proved that $\ell(n(\Theta_+ + \Theta_-)) = 4n^2$, the Hilbert series associated to this divisor is

$$HS_{\Theta_+ + \Theta_-}(t) = \frac{1 + t + 7t^2 - t^3}{(1-t)^3} = 1 + \sum_{n=1}^{\infty} 4n^2 t^n.$$

We will now give an explicit description of this embedding.

Proposition 6.3.1. *The Jacobian of any genus two curve admits a projective embedding inside $\mathbb{P}(1^4, 2^6)$ as the intersection of 8 cubics and 22 quartics.*

Proof. As previously noted, we have a map $\varphi_{|2(\Theta_+ + \Theta_-)|} : \mathrm{Jac}(\mathcal{C}) \hookrightarrow \mathbb{P}^{15}$. From the decomposition of the basis of $\mathcal{L}(2(\Theta_+ + \Theta_-))$ discussed in section 4.2, we deduce that this map factors as

$$\mathrm{Jac}(\mathcal{C}) \xrightarrow{\phi} \mathbb{P}(1^4, 2^6) \xrightarrow{\psi} \mathbb{P}^{15},$$

where ϕ is given by the projectivisation of the elements $\{k_1, \ldots, k_4, b_1, \ldots, b_6\}$, and ψ is the map constructed by considering all degree two polynomials in $\mathbb{P}(1^4, 2^6)$. We now examine the image of ϕ. According to the description by Flynn [Fly90], the space of 72 quadrics defining the Jacobian in $\mathbb{P}^{15}$ decomposes into:

- A 20-dimensional subspace formed by the quadrics arising from the Veronese embedding associated to the degree-two Veronese map $\mathbb{P}(1^4) \to \mathbb{P}(2^{10})$.
- A 30-dimensional subspace coming from the 8 relations that exist among the $b_i k_j$ with $1 \leq i \leq 6$, $1 \leq j \leq 4$. Multiplying each of these relations by $k_1, \ldots, k_4$ yields 32 relations among the $b_i k_j k_r$, of which 2 are linearly dependent.
- A 22-dimensional subspace of quadrics, where one comes from the quartic defining the Kummer surface in $\mathbb{P}^3$, and the remaining 21 express all monomials of the form $b_i b_j$ in terms of quartic polynomials in the k_s.

Therefore, the pullback of the Jacobian through ψ gives a variety $X_{3^8,4^{22}} \subseteq \mathbb{P}(1^4, 2^6)$ defined by 8 cubics and 22 quartics. Using these equations, we can compute the Hilbert series of this variety in Magma and verify that $HS_{\Theta_+ + \Theta_-}(t) = HS_{X_{3^8,4^{22}}}(t)$. Since there is a closed immersion $X_{3^8,4^{22}} \hookrightarrow \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathcal{L}(n(\Theta_+ + \Theta_-))\right)$ and the two Hilbert series agree, this map is an isomorphism. □

Remark 6.3.2. *In characteristic two, a similar argument to the proof of proposition 6.3.1 using the coordinates $\{v_1, \dots, v_6\}$ from section 4.4.3, instead of $\{b_1, \dots, b_6\}$, also yields a model of the Jacobian as a subvariety $X_{3^8,4^{22}} \subseteq \mathbb{P}(1^4, 2^6)$.*

6.3.1 The quotient by C_3

Let $\mathcal{C}$ be a genus two curve with $\mathrm{Aut}(\mathcal{C}) = D_6$ that admits an action of the subgroup C_3 generated by the automorphism $\tau_3 \coloneqq \tau_6^2$, as described in subsection 6.2.1, and let $\mathrm{Jac}(\mathcal{C})$ be its Jacobian. We saw that the action of τ_3 on $\mathcal{C}$ induces a symplectic action $\tilde{\tau}_3$ on $\mathrm{Jac}(\mathcal{C})$. We consider a linearisation of the line bundle $\mathcal{O}_{\mathrm{Jac}(\mathcal{C})}(1)$ so that the action of C_3 lifts to a linear action on $\mathcal{L}(n(\Theta_+ + \Theta_-))$ for all n. To construct such a linearisation, we verify that $\tilde{\tau}_3$ acts linearly on the basis $\{k_1, \dots, k_4, b_1, \dots, b_6\}$ of $\bigoplus_{n \geq 0} \mathcal{L}(n(\Theta_+ + \Theta_-))$:

$$\begin{aligned} \tilde{\tau}_3\colon \quad & (k_1, k_2, k_3, k_4) \longmapsto (k_1, \zeta_3 k_2, \zeta_3^2 k_3, \zeta_3 k_4), \\ & (b_1, b_2, b_3, b_4, b_5, b_6) \longmapsto (\zeta_3^2 b_1, b_2, \zeta_3 b_3, \zeta_3^2 b_4, b_5, \zeta_3 b_6), \end{aligned}$$

where ζ_3 is a primitive third root of unity.

By twisting the linearisation on $\mathcal{O}_{\mathrm{Jac}(\mathcal{C})}(1)$ by the three distinct characters of C_3, we obtain three different linearisations.

In the linearisation twisted by the character ζ_3^2, the ring of invariants is generated by two invariants of degree 1, three invariants of degree 2, and four invariants of degree 3. Therefore, the quotient admits an embedding as a codimension 6 subvariety of $\mathbb{P}(1^2, 2^3, 3^4)$. The Hilbert series of this embedding is

$$HS(t) = \frac{(1+t)(1-t+4t^2-t^3+t^4)}{(1-t)^2(1-t^3)}.$$

According to the graded ring database, this Hilbert series corresponds to the family of K3 surfaces with label 11468 [ABR02, Bro07].

If we instead choose either of the other two linearisations, the ring of invariants is generated by one invariant of degree 1, four of degree 2, and four of degree 3. Hence, the quotient admits an embedding as a codimension 11 subvariety of $\mathbb{P}(1, 2^4, 3^9)$. The Hilbert series of this embedding is

$$HS(t) = \frac{(1+t)(1-2t+6t^2-2t^3+t^4)}{(1-t)^2(1-t^3)},$$

which corresponds to the family of K3 surfaces with label 8049 in the graded ring database [ABR02, Bro07].

Regardless of the choice of linearisation, one can verify that the singular locus of $\mathrm{Kum}_{C_3}(\mathrm{Jac}(\mathcal{C}))$ consists of nine singularities of type A_2. One is O and the others arise as the images under the quotient map of the fixed points of $\mathrm{Jac}(\mathcal{C})$ under the C_3-action.

These elements are divisors of the form $(P)+(Q)-K_{\mathcal{C}}$, where P and Q are fixed points of the automorphism

$$\tau_6^2\colon \begin{cases} x \mapsto \zeta_3 x, \\ y \mapsto y \end{cases}$$

on $\mathcal{C}$, therefore, $P, Q \in \{0_+, 0_-, \infty_+, \infty_-\}$, where

$$0_+ \coloneqq \left(0, \tfrac{1}{2}(-1+\sqrt{1-4t})\right), \qquad 0_- \coloneqq \left(0, \tfrac{1}{2}(-1-\sqrt{1-4t})\right).$$

These points form a subgroup isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2 < \mathrm{Jac}(\mathcal{C})[3]$.

6.4 Computing embeddings for the quotients by the actions of groups of even order

As previously mentioned, whenever $G = \mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ and $G \neq C_3$, $\langle \iota_h \rangle \trianglelefteq G$ and, therefore, the map $\mathrm{Jac}(\mathcal{C}) \to \mathrm{Kum}_G(\mathrm{Jac}(\mathcal{C}))$ factors through $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$, where the map $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) \to \mathrm{Kum}_G(\mathrm{Jac}(\mathcal{C}))$ is the quotient by the induced action of $H = G/C_2$ on $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$.

6.4.1 The quotient by C_4

Let $\mathcal{C}$ be a curve with $\mathrm{Aut}(\mathcal{C}) = D_4$, which admits an action of the subgroup C_4 generated by the symplectic automorphism τ_4. In this case, $G/\langle \iota_h \rangle = C_2$ and the induced action of τ_4 on $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ is given by the automorphism of order two

$$\begin{aligned} \tau_4|_{\mathrm{Kum}}\colon \quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) \\ [k_1 : k_2 : k_3 : k_4] &\longmapsto [k_1 : -k_2 : k_3 : -k_4] \end{aligned}$$

In subsection 1.5.5, we explained how to construct a linearisation on $\mathcal{O}_{\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))}(1)$ associated to this map. We can construct a different linearisation by twisting it by the character -1. Either way, computing the invariant functions with respect to either of these linearisations, we can show that

$$\mathrm{Kum}_{C_4}(\mathrm{Jac}(\mathcal{C})) = \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))/\langle \tau_4|_{\mathrm{Kum}} \rangle$$

admits a model as the complete intersection of a quartic and a sextic $X_{4,6}$ inside of $\mathbb{P}(1^2, 2^3)$.

This surface has four A_3 singularities and six A_1 singularities. As we have seen, these correspond to the image of points in $\mathrm{Jac}(\mathcal{C})[2]$. Recall that the non-trivial elements in $\mathrm{Jac}(\mathcal{C})[2]$ are always of the form $(P)+(Q)-K_{\mathcal{C}}$ where $P \neq Q$ are Weierstrass points of $\mathcal{C}$. The Weierstrass points of $\mathcal{C}$ are $\{(0,0), \infty, (\alpha, 0), (-\alpha, 0), (\beta, 0), (-\beta, 0)\}$ where $\pm\alpha$ and $\pm\beta$ are the roots of $x^4 + x^2 + t$.

The A_3 singularities correspond to the subgroup $(\mathbb{Z}/2\mathbb{Z})^2 < \mathrm{Jac}(\mathcal{C})[2]$ whose non-trivial elements are

$$\begin{aligned} D_1 &= ((0,0)) + (\infty) - K_{\mathcal{C}}, \\ D_2 &= ((\alpha,0)) + ((-\alpha,0)) - K_{\mathcal{C}}, \\ D_3 &= ((\beta,0)) + ((-\beta,0)) - K_{\mathcal{C}}. \end{aligned}$$

The other six singularities correspond to the other twelve 2-torsion points, which are identified 2-to-1 in the quotient.

6.4.2 The quotient by C_6

Let $\mathcal{C}$ be a curve with $\mathrm{Aut}(\mathcal{C}) = D_6$, which admits an action of the subgroup C_6 generated by the symplectic automorphism τ_6. Then, $G/\langle \iota_h \rangle = C_3$ and the induced action of τ_6 on $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ is given by the automorphism of order three

$$\begin{aligned} \tau_6|_{\mathrm{Kum}}\colon \quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) \\ [k_1 : k_2 : k_3 : k_4] &\longmapsto [\zeta_3^2 k_1 : k_2 : \zeta_3 k_3 : k_4] \end{aligned}$$

There are three possible linearisations on $\mathcal{O}_{\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))}(1)$, corresponding to the three characters of C_3.

Computing the invariant functions with respect to the standard linearisation, we can check that

$$\mathrm{Kum}_{C_6}(\mathrm{Jac}(\mathcal{C})) = \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))/\langle \tau_6|_{\mathrm{Kum}} \rangle$$

admits a model as the complete intersection of a quartic and a sextic $X_{4,6}$ inside of $\mathbb{P}(1^2, 2, 3^2)$.

If we compute the quotient with respect to the other two twisted linearisations, the models that we obtain are varieties of codimension five inside of $\mathbb{P}(1, 2^2, 3^5)$ (label 5705 in the graded ring database).

The surface $\mathrm{Kum}_{C_6}(\mathrm{Jac}(\mathcal{C}))$ has one A_5, four A_2 and five A_1 singularities. The A_5 singularity corresponds to the image of the identity in $\mathrm{Jac}(\mathcal{C})$, the A_2 singularities correspond to the points that we described in section 6.3.1, which reduce 2-to-1 in the quotient, and the A_1 singularities correspond to the image of points in $\mathrm{Jac}(\mathcal{C})[2]$, which are contracted 3-to-1 in the quotient.

6.4.3 The quotient by Q_{12}

If $\mathcal{C}$ is the curve with $\mathrm{Aut}(\mathrm{Jac}(\mathcal{C})) = C_3 \rtimes D_4$, $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C})) = Q_{12}$ and the non-proper subgroups of $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ are C_2, C_3, C_4 and C_6, which all give rise of examples of generalised Kummer surfaces that we have already seen.

When $G = Q_{12}$, then $G/\langle\iota_h\rangle = S_3$. The elements in $\mathrm{Aut}(\mathrm{Jac}(\mathcal{C}))$ that generate this group are the ones corresponding to the automorphisms $\tau_2\tau_6, \tau_6^2 \in \mathrm{Aut}(\mathcal{C})$, which have orders four and three. The induced actions on $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$ are given by the automorphisms

$$\begin{aligned}(\tau_2\tau_6)|_{\mathrm{Kum}}\colon \quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))\\ [k_1 : k_2 : k_3 : k_4] &\longmapsto [\zeta_3 k_3 : -k_2 : \zeta_3^2 k_1 : k_4]\end{aligned}$$

$$\begin{aligned}\tau_6^2|_{\mathrm{Kum}}\colon \quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))\\ [k_1 : k_2 : k_3 : k_4] &\longmapsto [\zeta_3^2 k_1 : k_2 : \zeta_3 k_3 : k_4]\end{aligned}$$

which have orders two and three, respectively. As before, depending on the choice of linearisation, there are multiple models for the quotient surface. We will only describe the model for the standard linearisation on $\mathcal{O}_{\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))}(1)$, which induces an embedding of

$$\mathrm{Kum}_{Q_{12}}(\mathrm{Jac}(\mathcal{C})) = \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))/\langle(\tau_2\tau_6)|_{\mathrm{Kum}}, \tau_6^2|_{\mathrm{Kum}}\rangle$$

inside $\mathbb{P}(1, 2^2, 3, 4)$ as the intersection $X_{4,8}$ of two hypersurfaces of degrees four and eight.

The surface $\mathrm{Kum}_{Q_{12}}(\mathrm{Jac}(\mathcal{C}))$ has one D_5, three A_3, two A_2 and one A_1 singularity. The D_5 singularity is the image of the identity in $\mathrm{Jac}(\mathcal{C})$. The three A_3 are the images of the nine non-trivial points in $\mathrm{Jac}(\mathcal{C})[2]$ that are fixed by the actions of any of the three subgroups of order four: $\langle\tau_2\tau_6\rangle$, $\langle\tau_2\tau_6^3\rangle$ and $\langle\tau_2\tau_6^5\rangle$. The two A_2 singularities correspond to the images of the points

$$D_1 = ((0,1)) + (\infty_+) - K_{\mathcal{C}}, \qquad D_2 = 2(\infty_+) - K_{\mathcal{C}},$$

which generate a subgroup $(\mathbb{Z}/3\mathbb{Z})^2 < \mathrm{Jac}(\mathcal{C})[3]$ whose non-trivial elements map 4-to-1 into the A_2 singularities. The A_1 singularity corresponds to the image of any of the six other points of $\mathrm{Jac}(\mathcal{C})[2]$, which reduce 6-to-1 on the quotient.

6.4.4 The quotient by Q_8

The curve $\mathcal{C}$ with $\mathrm{Aut}(\mathcal{C}) = \mathrm{GL}_2(\mathbb{F}_3)$ satisfies that $\mathrm{Aut}(\mathrm{Jac}(\mathcal{C})) = \mathrm{SL}_2(\mathbb{F}_3)$ and the non-proper subgroups of $\mathrm{Aut}_s(\mathrm{Jac}(\mathcal{C}))$ are C_2, C_3, C_4, C_6 and Q_8.

When $G = Q_8$, then $G/\langle\iota_h\rangle = C_2^2$. The elements in $\mathrm{Aut}(\mathrm{Jac}(\mathcal{C}))$ that generate this group are the ones corresponding to the automorphisms $\sigma_3\tau_3, \tau_3\sigma_3^{-1}\tau_3 \in \mathrm{Aut}(\mathcal{C})$ of order four:

$$\sigma_3\tau_3\colon \begin{cases} x \mapsto -x, \\ y \mapsto it, \end{cases} \qquad \tau_3\sigma_3^{-1}\tau_3\colon \begin{cases} x \mapsto -\frac{1}{x}, \\ y \mapsto -\frac{y}{x^3}. \end{cases}$$

They induce the involutions on $\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))$:

$$\begin{aligned}(\sigma_3\tau_3)|_{\mathrm{Kum}}\colon\quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))\\ [k_1:k_2:k_3:k_4] &\longmapsto [k_1:-k_2:k_3:-k_4]\end{aligned}$$

$$\begin{aligned}(\tau_3\sigma_3^{-1}\tau_3)|_{\mathrm{Kum}}\colon\quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))\\ [k_1:k_2:k_3:k_4] &\longmapsto [k_3:-k_2:k_1:k_4]\end{aligned}$$

Computing the invariant functions with respect to the standard linearisation on $\mathcal{O}_{\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))}(1)$, we can show that

$$\mathrm{Kum}_{Q_8}(\mathrm{Jac}(\mathcal{C})) = \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))/\langle(\sigma_3\tau_3)|_{\mathrm{Kum}},(\tau_3\sigma_3^{-1}\tau_3)|_{\mathrm{Kum}}\rangle$$

admits a model as the complete intersection of a quartic and a sextic $X_{4,6}\subset\mathbb{P}(1,2^3,3)$.

This surface has four D_4 and three A_1 singularities. The D_4 singularities correspond to the images under the quotient map of the origin and the following three elements of $\mathrm{Jac}(\mathcal{C})[2]$:

$$\begin{aligned}D_1 &= ((0,0))+(\infty)-K_{\mathcal{C}},\\ D_2 &= ((1,0))+((-1,0))-K_{\mathcal{C}},\\ D_3 &= ((i,0))+((-i,0))-K_{\mathcal{C}}.\end{aligned}$$

The other twelve points of $\mathrm{Jac}(\mathcal{C})[2]$ reduce 4-to-1 into the three A_1 singularities.

6.4.5 The quotient by $\mathrm{SL}_2(\mathbb{F}_3)$

When $G=\mathrm{SL}_2(\mathbb{F}_3)$, then $G/\langle\iota_h\rangle = A_4$. The elements $\tau_3,\sigma_3\in\mathrm{Aut}(\mathcal{C})$ induce the automorphisms of $\mathrm{Kum}(\mathcal{C})$.

$$\begin{aligned}\tau_3|_{\mathrm{Kum}}\colon\quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))\\ [k_1:k_2:k_3:k_4] &\longmapsto \left[\tfrac{i}{2}(k_1+k_2+k_3):-k_1+k_3:\tfrac{i}{2}(-k_1+k_2-k_3):k_4\right]\end{aligned}$$

$$\begin{aligned}\sigma_3|_{\mathrm{Kum}}\colon\quad \mathrm{Kum}(\mathrm{Jac}(\mathcal{C})) &\longrightarrow \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))\\ [k_1:k_2:k_3:k_4] &\longmapsto \left[\tfrac{1}{2}(ik_1-k_2-ik_3):i(k_1+k_3):\tfrac{1}{2}(ik_1+k_2-ik_3):k_4\right]\end{aligned}$$

Computing the invariant functions with respect to the linearisation on $\mathcal{O}_{\mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))}(1)$, we can check that

$$\mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}(\mathrm{Jac}(\mathcal{C})) = \mathrm{Kum}(\mathrm{Jac}(\mathcal{C}))/\langle\tau_3|_{\mathrm{Kum}},\sigma_3|_{\mathrm{Kum}}\rangle$$

admits a model as the intersection $X_{4,12}$ of two hypersurfaces of degree four and twelve inside $\mathbb{P}(1,2,3,4,6)$.

$\mathrm{Kum}_{\mathrm{SL}_2(\mathbb{F}_3)}$ has one E_6, one D_4, four A_2 and one A_1 singularity. The E_6 singularity is the image of the identity in $\mathrm{Jac}(\mathcal{C})$, and the D_4 correspond to the image of any of the divisors $D_i \in \mathrm{Jac}(\mathcal{C})$ that we saw in subsection 6.4.4, which contract 3-to-1 in the quotient. The A_1 singularity corresponds to the rest of the points in $\mathrm{Jac}(\mathcal{C})[2]$, which contract 12-to-1 in the quotient.

As for the four A_2 singularities, up to conjugacy, there are four copies of C_3 inside of $\mathrm{SL}_2(\mathbb{F}_3)$: $\langle\tau_3\rangle$, $\langle\sigma_3\rangle$, $\langle\tau_3\sigma_3\tau_3^{-1}\rangle$ and $\langle\tau_3^{-1}\sigma_3\tau_3\rangle$. Each of these copies of C_3 fixes a different subgroup $(\mathbb{Z}/3\mathbb{Z})^2 < \mathrm{Jac}(\mathcal{C})[3]$ and no non-trivial elements of $\mathrm{Jac}(\mathcal{C})[3]$ are fixed than more than one copy. Therefore there are precisely 32 non-trivial elements of $\mathrm{Jac}(\mathcal{C})[3]$ fixed by a copy of C_3. These reduce 8-to-1 to the four A_2 singularities in the quotient.

6.5 The cases where $p \mid |G|$

If p divides the order of G, then combining propositions 2.2.10 and 6.2.3, we see that the possible subgroups G for which $\mathrm{Jac}(\mathcal{C})/G$ may be a K3 surface are as follows:

- If $p = 2$, then G can be either C_2 or C_6.
- If $p = 3$, then G can be C_3, C_6, or $\mathrm{SL}_2(\mathbb{F}_3)$.
- If $p = 5$, then the only possible G is C_5.

If $p = 2$, the case where $G = \langle\iota_h\rangle$ has already been studied in chapter 4. If G is generated by any other involution τ_2, then the action is not rigid. This is because the subvariety

$$W = \{D \in \mathrm{Jac}(\mathcal{C}) : D = (P) + (\sigma(P)) - K_{\mathcal{C}},\ \text{where } P \in \mathcal{C}\}$$

is one-dimensional and all points in W are fixed by σ.

Hence, by applying theorem 5.5.1 to determine the possible p-ranks of the families of curves with automorphism groups D_4, D_6, $\mathrm{GL}_2(\mathbb{F}_3)$, and $\mathrm{SL}_2(\mathbb{F}_5)$, we conclude that the only cases left to study correspond to the following quotients:

- The action of C_6 in characteristic two when $\mathrm{Jac}(\mathcal{C})$ is ordinary.
- The action of C_3 in characteristic three when $\mathrm{Jac}(\mathcal{C})$ is ordinary.
- The action of C_6 in characteristic three when $\mathrm{Jac}(\mathcal{C})$ is ordinary.
- The action of $\mathrm{SL}_2(\mathbb{F}_3)$ in characteristic three when $\mathrm{Jac}(\mathcal{C})$ is ordinary.
- The action of C_5 in characteristic five when $\mathrm{Jac}(\mathcal{C})$ is supersingular.

It is expected that, except for the last case, all quotients should give rise to generalised Kummer surfaces, and their singular models should have the configuration of singularities that we saw in table 2.3.

A Appendix

A.1 Supplementary equations

A.1.1 Basis of $\mathcal{L}(\Theta_+ + \Theta_-)$ in characteristic two

$$\bar{k}_1 = 1, \qquad \bar{k}_2 = x_1 + x_2,$$
$$\bar{k}_3 = x_1 x_2, \qquad \bar{k}_4 = \frac{S(x_1, x_2) + y_2 g(x_1) + y_1 g(x_2)}{(x_1 + x_2)^2},$$

where

$$S(u, v) = f_0 + f_1(u + v) + f_3 uv(u + v)^2 + f_5 u^2 v^2 (u + v)^2.$$

A.1.2 Reduction of the odd elements of $\mathcal{L}(\Theta_+ + \Theta_-)$ in characteristic two

$$\begin{aligned}
\bar{b}_1 &= \frac{g(x_1) + g(x_2)}{x_1 + x_2}, \\
\bar{b}_2 &= \frac{x_2 g(x_1) + x_1 g(x_2)}{x_1 + x_2}, \\
\bar{b}_3 &= \frac{x_2^2 g(x_1) + x_1^2 g(x_2)}{x_1 + x_2}, \\
\bar{b}_4 &= \frac{y_2 g(x_1) + y_1 g(x_2)}{x_1 + x_2}, \\
\bar{b}_5 &= \frac{(g_1 + g_2(x_1 + x_2) + g_3 x_1 x_2)(y_2 g(x_1) + y_1 g(x_2))}{(x_1 + x_2)^2} + \frac{T(x_1, x_2)}{x_1 + x_2}, \\
\bar{b}_6 &= \frac{g_3(g_0 g_3 + g_2^2(x_1 + x_2) + g_2 g_3 x_1 x_2)(y_2 g(x_1) + y_1 g(x_2))}{(x_1 + x_2)^2} + \frac{g_3^2 R(x_1, x_2)}{x_1 + x_2}.
\end{aligned}$$

Here $T(u,v)$ and $R(u,v)$ are the following bivariate polynomials:

$$\begin{aligned}
T(u,v) &= f_1g_1 + (f_3g_1 + f_1g_3)uv + (f_5g_1 + f_3g_3)u^2v^2 + f_5g_3u^3v^3 + (f_3g_0 + f_1g_2)(u+v) \\
&\quad + g_0g_2g_3uv(u+v) + (f_5g_2 + g_2^2g_3)u^2v^2(u+v) + f_1g_3(u+v)^2 + (f_5g_1 + g_1g_2g_3)uv(u+v)^2 \\
&\quad + (f_5g_0 + g_0g_2g_3)(u+v)^3, \\
R(u,v) &= f_1g_0 + (f_3g_0 + f_1g_2)uv + (f_5g_0 + f_3g_2)u^2v^2 + f_5g_2u^3v^3 + (f_1g_1 + g_0^2g_2)(u+v) \\
&\quad + (g_0g_2^2 + g_0g_1g_3)uv(u+v) + (f_5g_1 + f_3g_3 + g_1g_2g_3)u^2v^2(u+v) + f_1g_2(u+v)^2 \\
&\quad + (f_5g_0 + g_1^2g_3 + g_0g_2g_3)uv(u+v)^2 + (f_1g_3 + g_0g_1g_3)(u+v)^3.
\end{aligned}$$

A.1.3 Basis of odd functions expressed in terms of the basis of even functions

$$\begin{aligned}
\bar{b}_1 &= g_1\bar{k}_1^2 + g_2\bar{k}_1\bar{k}_2 + g_3\bar{k}_2^2 + g_3\bar{k}_1\bar{k}_3, \\
\bar{b}_2 &= g_0\bar{k}_1^2 + g_2\bar{k}_1\bar{k}_3 + g_3\bar{k}_2\bar{k}_3, \\
\bar{b}_3 &= g_0\bar{k}_1\bar{k}_2 + g_1\bar{k}_1\bar{k}_3 + g_3\bar{k}_3^2, \\
\bar{b}_4 &= f_1\bar{k}_1^2 + f_3\bar{k}_1\bar{k}_3 + f_5\bar{k}_3^2 + \bar{k}_2\bar{k}_4, \\
\bar{b}_5 &= f_3g_0\bar{k}_1^2 + f_1g_3\bar{k}_1\bar{k}_2 + (f_5g_0 + g_0g_2g_3)\bar{k}_2^2 + (f_3g_2 + g_0g_2g_3)\bar{k}_1\bar{k}_3 + (f_5g_1 + g_1g_2g_3)\bar{k}_2\bar{k}_3 + g_2^2g_3\bar{k}_3^2 + g_1\bar{k}_1\bar{k}_4 \\
&\quad + g_2\bar{k}_2\bar{k}_4 + g_3\bar{k}_3\bar{k}_4, \\
\bar{b}_6 &= (f_1g_2^2g_3 + f_1g_1g_3^2 + g_0^2g_2g_3^2)\bar{k}_1^2 + f_1g_2g_3^2\bar{k}_1\bar{k}_2 + (f_1g_3^3 + g_0g_1g_3^3)\bar{k}_2^2 + (f_3g_2^2g_3 + g_0g_2^2g_3^2 + g_0g_1g_3^3)\bar{k}_1\bar{k}_3 \\
&\quad + (f_5g_0g_3^2 + g_1^2g_3^3 + g_0g_2g_3^3)\bar{k}_2\bar{k}_3 + (f_5g_2^2g_3 + f_5g_1g_3^2 + f_3g_3^3 + g_1g_2g_3^3)\bar{k}_3^2 + g_0g_3^2\bar{k}_1\bar{k}_4 + g_2^2g_3\bar{k}_2\bar{k}_4 + g_2g_3^2\bar{k}_3\bar{k}_4.
\end{aligned}$$

A.1.4 Quartic polynomial defining the equation of the Kummer in $\mathbb{P}^3$ in characteristic two

$$\begin{aligned}
q(\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4) &= (f_1^2 + f_2g_0^2 + f_1g_0g_1 + f_0g_1^2)\bar{k}_1^4 + (f_3g_0^2 + f_1g_0g_2)\bar{k}_1^3\bar{k}_2 + (f_4g_0^2 + f_0g_2^2 + f_1g_0g_3)\bar{k}_1^2\bar{k}_2^2 \\
&\quad + f_5g_0^2\bar{k}_1\bar{k}_2^3 + (f_6g_0^2 + f_0g_3^2)\bar{k}_2^4 + (f_3g_0g_1 + f_1g_1g_2 + f_1g_0g_3)\bar{k}_1^3\bar{k}_3 + (f_5g_0^2 + f_3g_0g_2)\bar{k}_1^2\bar{k}_2\bar{k}_3 \\
&\quad + (f_1g_2^2 + f_1g_1g_3)\bar{k}_1^2\bar{k}_2\bar{k}_3 + f_3g_0g_3\bar{k}_1\bar{k}_2^2\bar{k}_3 + f_1g_3^2\bar{k}_2^3\bar{k}_3 + (f_3^2 + f_6g_0^2 + f_5g_0g_1 + f_4g_1^2)\bar{k}_1^2\bar{k}_3^2 \\
&\quad + (f_3g_1g_2 + f_2g_2^2 + f_3g_0g_3 + f_1g_2g_3 + f_0g_3^2)\bar{k}_1^2\bar{k}_3^2 + (f_5g_1^2 + f_5g_0g_2 + f_3g_1g_3 + f_1g_3^2)\bar{k}_1\bar{k}_2\bar{k}_3^2 \\
&\quad + (f_6g_1^2 + f_5g_0g_3 + f_2g_3^2)\bar{k}_2^2\bar{k}_3^2 + (f_5g_1g_2 + f_5g_0g_3 + f_3g_2g_3)\bar{k}_1\bar{k}_3^3 + (f_5g_1g_3 + f_3g_3^2)\bar{k}_2\bar{k}_3^3 \\
&\quad + (f_5^2 + f_6g_2^2 + f_5g_2g_3 + f_4g_3^2)\bar{k}_3^4 + g_0^2\bar{k}_1^3\bar{k}_4 + g_0g_1\bar{k}_1^2\bar{k}_2\bar{k}_4 + g_0g_2\bar{k}_1\bar{k}_2^2\bar{k}_4 + g_0g_3\bar{k}_2^3\bar{k}_4 \\
&\quad + g_1^2\bar{k}_1^2\bar{k}_3\bar{k}_4 + (g_1g_2 + g_0g_3)\bar{k}_1\bar{k}_2\bar{k}_3\bar{k}_4 + g_1g_3\bar{k}_2^2\bar{k}_3\bar{k}_4 + g_2^2\bar{k}_1\bar{k}_3^2\bar{k}_4 + g_2g_3\bar{k}_2\bar{k}_3^2\bar{k}_4 + g_3^2\bar{k}_3^3\bar{k}_4 + \bar{k}_2^2\bar{k}_4^2.
\end{aligned}$$

A.1.5 Quadratics defining the equations of the Kummer in $\mathbb{P}^5$ in characteristic two

$$
\begin{aligned}
c_1(\bar{b}_1,\bar{b}_2,\bar{b}_3,\bar{b}_4,\bar{b}_5,\bar{b}_6) &= (f_1g_3^2+g_0g_1g_3^2)\bar{b}_1^2+g_1^2g_3^2\bar{b}_1\bar{b}_2+(f_3g_3^2+g_1g_2g_3^2)\bar{b}_2^2+g_1g_2g_3^2\bar{b}_1\bar{b}_3+g_2^2g_3^2\bar{b}_2\bar{b}_3\\
&\quad+f_5g_3^2\bar{b}_3^2+g_2^2g_3\bar{b}_1\bar{b}_4+g_3^3\bar{b}_3\bar{b}_4+g_3^2\bar{b}_2\bar{b}_5+\bar{b}_1\bar{b}_6,\\
c_2(\bar{b}_1,\bar{b}_2,\bar{b}_3,\bar{b}_4,\bar{b}_5,\bar{b}_6) &= (f_6g_0^2+f_0g_3^2)\bar{b}_1^2+g_0g_1g_3^2\bar{b}_1\bar{b}_2+(f_6g_1^2+f_2g_3^2+g_1^2g_3^2+g_0g_2g_3^2)\bar{b}_2^2+g_0g_2g_3^2\bar{b}_1\bar{b}_3\\
&\quad+(f_6g_2^2+f_4g_3^2+g_2^2g_3^2)\bar{b}_3^2+g_0g_3^2\bar{b}_1\bar{b}_4+(g_2^2g_3+g_1g_3^2)\bar{b}_2\bar{b}_4+g_2g_3^2\bar{b}_3\bar{b}_4+g_3^2\bar{b}_4^2+g_3^2\bar{b}_3\bar{b}_5+\bar{b}_2\bar{b}_6,\\
c_3(\bar{b}_1,\bar{b}_2,\bar{b}_3,\bar{b}_4,\bar{b}_5,\bar{b}_6) &= (f_5g_0^2+g_0^2g_2g_3)\bar{b}_1^2+(f_5g_1^2+g_1^2g_2g_3+f_1g_3^2)\bar{b}_2^2+g_0g_1g_3^2\bar{b}_1\bar{b}_3+(g_1^2g_3^2+g_0g_2g_3^2)\bar{b}_2\bar{b}_3\\
&\quad+(f_5g_2^2+g_2^3g_3+f_3g_3^2+g_1g_2g_3^2)\bar{b}_3^2+g_0g_2g_3\bar{b}_1\bar{b}_4+(g_1g_2g_3+g_0g_3^2)\bar{b}_2\bar{b}_4+g_1g_3^2\bar{b}_3\bar{b}_4\\
&\quad+g_0g_3\bar{b}_1\bar{b}_5+g_1g_3\bar{b}_2\bar{b}_5+g_2g_3\bar{b}_3\bar{b}_5+\bar{b}_3\bar{b}_6.
\end{aligned}
$$

A.1.6 Equations defining the rational map $Y \dashrightarrow X$ in characteristic two

The rational map is given by

$$
\begin{aligned}
Y &\dashrightarrow X\\
[\bar{b}_1:\cdots:\bar{b}_6] &\longmapsto [p_1:p_2:p_3:p_4]
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= g_3^2g(x_1)g(x_2)\bar{k}_1 = g_3^2(\bar{b}_2^2+\bar{b}_1\bar{b}_3),\\
p_2 &= g_3^2g(x_1)g(x_2)\bar{k}_2 = g_3(g_0\bar{b}_1^2+g_1\bar{b}_2\bar{b}_1+g_2\bar{b}_3\bar{b}_1+g_3\bar{b}_2\bar{b}_3),\\
p_3 &= g_3^2g(x_1)g(x_2)\bar{k}_3 = g_3(g_1\bar{b}_2^2+g_0\bar{b}_1\bar{b}_2+g_2\bar{b}_3\bar{b}_2+g_3\bar{b}_3^2),\\
p_4 &= g_3^2g(x_1)g(x_2)\bar{k}_4 = f_0g_3^2\bar{b}_1^2+f_2g_3^2\bar{b}_2^2+f_4g_3^2\bar{b}_3^2+f_1g_3^2\bar{b}_1\bar{b}_2+f_3g_3^2\bar{b}_2\bar{b}_3+f_5g_2g_3\bar{b}_3^2+f_5g_0g_3\bar{b}_1\bar{b}_3\\
&\qquad+f_5g_1g_3\bar{b}_2\bar{b}_3+f_6g_0^2\bar{b}_1^2+f_6g_1^2\bar{b}_2^2+f_6g_2^2\bar{b}_3^2+g_3^2\bar{b}_4^2.
\end{aligned}
$$

A.1.7 Equation of the Weddle surface in characteristic two

$$
\begin{aligned}
q(\bar{b}_1,\bar{b}_2,\bar{b}_3,\bar{b}_4) &= g_0(f_6g_0^2+f_0g_3^2)\bar{b}_1^4+(f_6g_0^2g_1+f_1g_0g_3^2+f_0g_1g_3^2)\bar{b}_1^3\bar{b}_2+(f_6g_0g_1^2+f_5g_0^2g_3+f_2g_0g_3^2)\bar{b}_1^2\bar{b}_2^2\\
&\quad+f_1g_1g_3^2\bar{b}_1^2\bar{b}_2^2+(f_6g_1^3+f_3g_0g_3^2+f_2g_1g_3^2)\bar{b}_1\bar{b}_2^3+g_3(f_5g_1^2+f_3g_1g_3+f_1g_3^2)\bar{b}_2^4+f_6g_0^2g_2\bar{b}_1^3\bar{b}_3\\
&\quad+(f_5g_0^2g_3+f_0g_2g_3^2)\bar{b}_1^3\bar{b}_3+g_3(f_6g_0^2+f_1g_2g_3+f_0g_3^2)\bar{b}_1^2\bar{b}_2\bar{b}_3+(f_6g_1^2g_2+f_5g_1^2g_3)\bar{b}_1\bar{b}_2^2\bar{b}_3\\
&\quad+(f_2g_2g_3^2+f_1g_3^3)\bar{b}_1\bar{b}_2^2\bar{b}_3+g_3(f_6g_1^2+f_3g_2g_3+f_2g_3^2)\bar{b}_2^3\bar{b}_3+(f_6g_0g_2^2+f_4g_0g_3^2+f_1g_3^3)\bar{b}_1^2\bar{b}_3^2\\
&\quad+(f_6g_1g_2^2+f_5g_0g_3^2+f_4g_1g_3^2)\bar{b}_1\bar{b}_2\bar{b}_3^2+f_5g_3(g_2^2+g_1g_3)\bar{b}_2^2\bar{b}_3^2+(f_6g_2^3+f_5g_2^2g_3)\bar{b}_1\bar{b}_3^3\\
&\quad+(f_4g_2g_3^2+f_3g_3^3)\bar{b}_1\bar{b}_3^3+g_3(f_6g_2^2+f_5g_2g_3+f_4g_3^2)\bar{b}_2\bar{b}_3^3+f_5g_3^3\bar{b}_3^4+g_0^2g_3^2\bar{b}_1^3\bar{b}_4+g_1^2g_3^2\bar{b}_1\bar{b}_2^2\bar{b}_4\\
&\quad+g_0g_2g_3^2\bar{b}_1\bar{b}_2^2\bar{b}_4+g_3^2(g_1g_2+g_0g_3)\bar{b}_2^3\bar{b}_4+g_0g_2g_3^2\bar{b}_1^2\bar{b}_3\bar{b}_4+g_3^2(g_1g_2+g_0g_3)\bar{b}_1\bar{b}_2\bar{b}_3\bar{b}_4\\
&\quad+g_3^2(g_2^2+g_1g_3)\bar{b}_2^2\bar{b}_3\bar{b}_4+g_1g_3^3\bar{b}_1\bar{b}_3^2\bar{b}_4+g_3^4\bar{b}_3^3\bar{b}_4+g_0g_3^2\bar{b}_1^2\bar{b}_4^2+g_1g_3^2\bar{b}_1\bar{b}_2\bar{b}_4^2+g_2g_3^2\bar{b}_1\bar{b}_3\bar{b}_4^2+g_3^3\bar{b}_2\bar{b}_3\bar{b}_4^2.
\end{aligned}
$$

A.1.8 Change of variables that connect with Katsura and Kondō's model for ordinary abelian surfaces

$$
\begin{aligned}
a_1 &= \alpha_1,\\
a_2 &= \alpha_2,\\
a_3 &= \alpha_3,\\
c_1 &= \frac{1}{\Delta_g},\\
c_2 &= \frac{g_3^4(\alpha_1+\alpha_3)^4(f_1+\alpha_2^2 f_3+\alpha_2^4 f_5+g_3(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)\beta_2)^2}{\Delta_g},\\
c_3 &= \frac{g_3^4(\alpha_1+\alpha_2)^4(f_1+\alpha_3^2 f_3+\alpha_3^4 f_5+g_3(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\beta_3)^2}{\Delta_g},\\
d_1 &= \frac{g_3^4(\alpha_2+\alpha_3)^4(f_1+\alpha_1^2 f_3+\alpha_1^4 f_5+g_3(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)\beta_1)^2}{\Delta_g},\\
d_2 &= \frac{1}{\Delta_g},\\
d_3 &= \frac{1}{\Delta_g}.
\end{aligned}
$$

where Δ_g is the discriminant of $g(x)$, $\Delta_g = g_3^4(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_3)^2(\alpha_2+\alpha_3)^2$.

A.1.9 Change of variables with Katsura and Kondō's model

$$
\begin{aligned}
X_1 &= (\alpha_2+\alpha_3)^2(\alpha_2\alpha_3 g_3^2(f_1+\alpha_1^2 f_3+\alpha_1^4 f_5+\alpha_1^5 g_3^2+\alpha_1^4\alpha_2 g_3^2+\alpha_1^4\alpha_3 g_3^2+\alpha_1^2\alpha_2^2\alpha_3 g_3^2+\alpha_1^2\alpha_2\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^2 g_3^2)\bar{b}_1\\
&\quad+g_3^2(\alpha_2 f_1+\alpha_3 f_1+\alpha_1^2\alpha_2 f_3+\alpha_1^2\alpha_3 f_3+\alpha_1^4\alpha_2 f_5+\alpha_1^4\alpha_3 f_5+\alpha_1^5\alpha_2 g_3^2+\alpha_1^4\alpha_2^2 g_3^2+\alpha_1^5\alpha_3 g_3^2+\alpha_1^3\alpha_2^2\alpha_3 g_3^2)\bar{b}_2\\
&\quad+g_3^2(\alpha_1^4\alpha_3^2 g_3^2+\alpha_1^3\alpha_2\alpha_3^2 g_3^2+\alpha_1^2\alpha_2^2\alpha_3^2 g_3^2+\alpha_1\alpha_2^3\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^3 g_3^2+\alpha_2^3\alpha_3^3 g_3^2)\bar{b}_2+g_3^2(f_1+\alpha_1^2 f_3+\alpha_1^4 f_5+\alpha_1^5 g_3^2)\bar{b}_3\\
&\quad+g_3^2(\alpha_1^3\alpha_2^2 g_3^2+\alpha_1^2\alpha_2^2\alpha_3 g_3^2+\alpha_1\alpha_2^3\alpha_3 g_3^2+\alpha_1^3\alpha_3^2 g_3^2+\alpha_1^2\alpha_2\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^2 g_3^2+\alpha_2^3\alpha_3^2 g_3^2+\alpha_1\alpha_2\alpha_3^3 g_3^2+\alpha_2^2\alpha_3^3 g_3^2)\bar{b}_3\\
&\quad+(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)^2 g_3^3\bar{b}_4+\alpha_1(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)g_3^2\bar{b}_5+(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)\bar{b}_6),\\
X_2 &= \alpha_1\alpha_3\bar{b}_1+(\alpha_1+\alpha_3)\bar{b}_2+\bar{b}_3,\\
X_3 &= \alpha_1\alpha_2\bar{b}_1+(\alpha_1+\alpha_2)\bar{b}_2+\bar{b}_3,\\
Y_1 &= \alpha_2\alpha_3\bar{b}_1+(\alpha_2+\alpha_3)\bar{b}_2+\bar{b}_3,\\
Y_2 &= (\alpha_1+\alpha_3)^2(\alpha_1\alpha_3 g_3^2(f_1+\alpha_2^2 f_3+\alpha_2^4 f_5+\alpha_1\alpha_2^4 g_3^2+\alpha_2^5 g_3^2+\alpha_1^2\alpha_2^2\alpha_3 g_3^2+\alpha_2^4\alpha_3 g_3^2+\alpha_1^2\alpha_2\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^2 g_3^2)\bar{b}_1\\
&\quad+g_3^2(\alpha_1 f_1+\alpha_3 f_1+\alpha_1\alpha_2^2 f_3+\alpha_2^2\alpha_3 f_3+\alpha_1\alpha_2^4 f_5+\alpha_2^4\alpha_3 f_5+\alpha_1^2\alpha_2^4 g_3^2+\alpha_1\alpha_2^5 g_3^2+\alpha_1^2\alpha_2^3\alpha_3 g_3^2+\alpha_2^5\alpha_3 g_3^2)\bar{b}_2\\
&\quad+g_3^2(\alpha_1^3\alpha_2\alpha_3^2 g_3^2+\alpha_1^2\alpha_2^2\alpha_3^2 g_3^2+\alpha_1\alpha_2^3\alpha_3^2 g_3^2+\alpha_2^4\alpha_3^2 g_3^2+\alpha_1^3\alpha_3^3 g_3^2+\alpha_1^2\alpha_2\alpha_3^3 g_3^2)\bar{b}_2+g_3^2(f_1+\alpha_2^2 f_3+\alpha_2^4 f_5+\alpha_2^5 g_3^2)\bar{b}_3\\
&\quad+g_3^2(\alpha_1^2\alpha_2^3 g_3^2+\alpha_1^3\alpha_2\alpha_3 g_3^2+\alpha_1^2\alpha_2^2\alpha_3 g_3^2+\alpha_1^3\alpha_3^2 g_3^2+\alpha_1^2\alpha_2\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^2 g_3^2+\alpha_2^3\alpha_3^2 g_3^2+\alpha_1^2\alpha_3^3 g_3^2+\alpha_1\alpha_2\alpha_3^3 g_3^2)\bar{b}_3\\
&\quad+(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)^2(\alpha_2+\alpha_3)g_3^3\bar{b}_4+\alpha_2(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)g_3^2\bar{b}_5+(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)\bar{b}_6),\\
Y_3 &= (\alpha_1+\alpha_2)^2(\alpha_1\alpha_2 g_3^2(f_1+\alpha_3^2 f_3+\alpha_3^4 f_5+\alpha_1^2\alpha_2^2\alpha_3 g_3^2+\alpha_1^2\alpha_2\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^2 g_3^2+\alpha_1\alpha_3^4 g_3^2+\alpha_2\alpha_3^4 g_3^2+\alpha_3^5 g_3^2)\bar{b}_1\\
&\quad+g_3^2(\alpha_1 f_1+\alpha_2 f_1+\alpha_1\alpha_3^2 f_3+\alpha_2\alpha_3^2 f_3+\alpha_1\alpha_3^4 f_5+\alpha_2\alpha_3^4 f_5+\alpha_1^3\alpha_2^3 g_3^2+\alpha_1^3\alpha_2^2\alpha_3 g_3^2+\alpha_1^2\alpha_2^3\alpha_3 g_3^2+\alpha_1^2\alpha_2^2\alpha_3^2 g_3^2)\bar{b}_2\\
&\quad+g_3^2(\alpha_1^2\alpha_2\alpha_3^3 g_3^2+\alpha_1\alpha_2^2\alpha_3^3 g_3^2+\alpha_1^2\alpha_3^4 g_3^2+\alpha_2^2\alpha_3^4 g_3^2+\alpha_1\alpha_3^5 g_3^2+\alpha_2\alpha_3^5 g_3^2)\bar{b}_2+g_3^2(f_1+\alpha_3^2 f_3+\alpha_3^4 f_5+\alpha_1^3\alpha_2^2 g_3^2)\bar{b}_3\\
&\quad+g_3^2(\alpha_1^2\alpha_2^3 g_3^2+\alpha_1^3\alpha_2\alpha_3 g_3^2+\alpha_1^2\alpha_2^2\alpha_3 g_3^2+\alpha_1\alpha_2^3\alpha_3 g_3^2+\alpha_1^2\alpha_2\alpha_3^2 g_3^2+\alpha_1\alpha_2^2\alpha_3^2 g_3^2+\alpha_1^2\alpha_3^3 g_3^2+\alpha_2^2\alpha_3^3 g_3^2+\alpha_3^5 g_3^2)\bar{b}_3\\
&\quad+(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)g_3^3\bar{b}_4+\alpha_3(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)g_3^2\bar{b}_5+(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\bar{b}_6).
\end{aligned}
$$

Bibliography

[ABR02] Selma Altınok, Gavin Brown, and Miles Reid, *Fano 3-folds, K3 surfaces and graded rings*, 2002, pp. 25–53. ↑23, 136

[Abr88] Victor A. Abrashkin, *Galois moduli of period p group schemes over a ring of Witt vectors*, Mathematics of the USSR-Izvestiya **31** (1988), no. 1, 1–46. ↑91

[Abr90] Victor A. Abrashkin, *Modular representations of the Galois group of a local field, and a generalization of the Shafarevich conjecture*, Mathematics of the USSR-Izvestiya **35** (1990), no. 3, 469–518. ↑91

[AFJR15] Chad Awtrey, Robin French, Peter Jakes, and Alan Russell, *Irreducible sextic polynomials and their absolute resolvents*, Minnesota Journal of Undergraduate Mathematics **1** (2015), no. 1. ↑62

[AH19] Jeffrey Achter and Everett Howe, *Hasse–Witt and Cartier–Manin matrices: A warning and a request*, 2019, pp. 1–18. ↑119

[Alb30] A. Adrian Albert, *On the Structure of Pure Riemann Matrices with Non-Commutative Multiplication Algebras*, Proceedings of the National Academy of Sciences **16** (1930), no. 4, 308–312. ↑14

[ALT14] Michela Artebani, Antonio Laface, and Damiano Testa, *On Büchi's K3 surface*, Mathematische Zeitschrift **278** (2014), no. 3-4, 1113–1131. ↑106

[Art66] Michael Artin, *On Isolated Rational Singularities of Surfaces*, American Journal of Mathematics **88** (1966), no. 1, 129. ↑6

[Art75] Michael Artin, *Coverings of the Rational Double Points in Characteristic p*, Complex analysis and algebraic geometry, 1975, pp. 11–22. ↑9, 54, 55

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma Algebra System I: The User Language*, Journal of Symbolic Computation **24** (1997), no. 3-4, 235–265. ↑65, 99

[Bro07] Gavin Brown, *A Database of Polarized K3 Surfaces*, Experimental Mathematics **16** (2007), no. 1, 7–20. ↑23, 136

[CF96] John William Scott Cassels and Eugene Victor Flynn, *Prolegomena to a Middle-brow Arithmetic of Curves of Genus 2*, Cambridge University Press, 1996. ↑20, 56, 57

[CQ07] Gabriel Cardona and Jordi Quer, *Curves of genus 2 with group of automorphisms isomorphic to D8 or D12*, Transactions of the American Mathematical Society **359** (2007), no. 6, 2831–2849. ↑100, 115, 117

[Dem21] Lassina Dembélé, *On the existence of abelian surfaces with everywhere good reduction*, Mathematics of Computation (2021). ↑91, 92, 95

[Dic01] Leonard Dickson, *Linear Groups: With an Exposition of the Galois Field Theory*, Dover Publications, New York, 1901. ↑33

[DK16] Lassina Dembélé and Abhinav Kumar, *Examples of abelian surfaces with everywhere good reduction*, Mathematische Annalen **364** (2016), no. 3-4, 1365–1392. ↑91, 95

[Dol23] Igor V. Dolgachev, *K3 surfaces of Kummer type in characteristic two*, preprint (2023). ↑83

[DS21] Andrzej Dabrowski and Mohammad Sadek, *Genus two curves with everywhere good reduction over quadratic fields*, preprint (2021). ↑91, 98

[Duq10] Sylvain Duquesne, *Traces of the group law on the Kummer surface of a curve of genus 2 in characteristic 2*, Mathematics in Computer Science **3** (2010), no. 2, 173–183. ↑70

[DV34] Patrick Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction (Part II.)*, Mathematical Proceedings of the Cambridge Philosophical Society **30** (1934), no. 4, 460–465. ↑6

[Ess24] Louis Esser, *Rational weighted projective hypersurfaces* (2024). ↑122

[Fly90] Eugene Victor Flynn, *The Jacobian and formal group of a curve of genus 2 over an arbitrary ground field*, Mathematical Proceedings of the Cambridge Philosophical Society **107** (1990), no. 3, 425–441. ↑67, 135

[Fly93] Eugene Victor Flynn, *The group law on the Jacobian of a curve of genus 2*, Journal für die reine und angewandte Mathematik (Crelles Journal) **1993** (1993), no. 439, 45–70. ↑61

[Fon77] Jean-Marc Fontaine, *Groupes p-divisibles sur les corps locaux*, Astérisque, Société mathématique de France, 1977 (fr). ↑18

[Fon85] Jean Marc Fontaine, *Il n'y a pas de variété abélienne sur Z*, Inventiones Mathematicae **81** (1985), no. 3, 515–538. ↑91

[Fon91] Jean Marc Fontaine, *Schemas propres et lisses sur les entiers*, Private communication (1991). ↑91

[FT19] Eugene Victor Flynn and Yan Bo Ti, *Genus Two Isogeny Cryptography*, 2019, pp. 286–306. ↑20

[FTvL12] Eugene Victor Flynn, Damiano Testa, and Ronald van Luijk, *Two-coverings of Jacobians of curves of genus 2*, Proceedings of the London Mathematical Society **104** (2012), no. 2, 387–429. ↑57, 59, 60, 66

[Fuj88] Akira Fujiki, *Finite Automorphism Groups of Complex Tori of Dimension Two*, Publications of the Research Institute for Mathematical Sciences **24** (1988), no. 1, 1–97. ↑51

[Gar17] Alice Garbagnati, *On K3 Surface Quotients of K3 or Abelian Surfaces*, Canadian Journal of Mathematics **69** (2017), no. 02, 338–372. ↑30

[GH94] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc., Hoboken, NJ, USA, 1994. ↑3

[Gra90] David Grant, *Formal groups in genus two*, Journal für die reine und angewandte Mathematik (Crelles Journal) **1990** (1990), no. 411, 96–121. ↑57

[HI80] Ki-ichiro Hashimoto and Tomoyoshi Ibukiyama, *On class numbers of positive definite binary quaternion Hermitian forms*, J. Fac. Sci., Univ. Tokyo, Sect. I A **27** (1980), 549–601 (English). ↑126

[Hos15] Victoria Hoskins, *Moduli problems and Geometric Invariant Theory*, Lecture notes (2015). ↑20, 22, 25

[HT15] Andrew Harder and Alan Thompson, *The Geometry and Moduli of K3 Surfaces*, 2015, pp. 3–43. ↑3

[IF00] Anthony R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, 2000, pp. 101–174. ↑23

[Igu58] Jun-Ichi Igusa, *Class Number of a Definite Quaternion with Prime Discriminant*, Proceedings of the National Academy of Sciences of the United States of America **44** (1958), no. 4, 312–314. ↑127

[Igu60] Jun-Ichi Igusa, *Arithmetic Variety of Moduli for Genus Two*, The Annals of Mathematics **72** (1960), no. 3, 612. ↑100, 102, 104

[IKO86] Tomoyoshi Ibukiyama, Toshiyuki Katsura, and Frans Oort, *Supersingular curves of genus two and class numbers*, Compositio Mathematica **2** (1986), no. 57, 127–152. ↑64, 120, 124, 126

[Kar24] Valentijn Karemaker, *Geometry and arithmetic of moduli spaces of abelian varieties in positive characteristic*, Lecture notes (2024). ↑12

[Kat78] Toshiyuki Katsura, *On Kummer surfaces in characteristic 2*, Proceedings on the International Symposium on Algebraic Geometry (1978), 525–542. ↑12, 64, 70

[Kat87] Toshiyuki Katsura, *Generalized Kummer surfaces and their unirationality in characteristic p*, Journal of the Faculty of Science of the University of Tokyo, Sect. IA, Math. **34** (1987), 1–41. ↑29, 31, 32, 40

[Keu97] Jong Hae Keum, *Automorphisms of Jacobian Kummer surfaces*, Compositio Mathematica **107** (1997), no. 3, 269–288. ↑63

[KK23] Toshiyuki Katsura and Shigeyuki Kondo, *Kummer surfaces and quadric line complexes in characteristic two*, preprint (2023). ↑57, 64

[KM24] Shigeyuki Kondo and Shigeru Mukai, *The automorphism groups of Kummer surfaces in characteristic two* (2024). ↑43

[KM98] Janos Kollár and Shigefumi Mori, *Birational Geometry of Algebraic Varieties*, Cambridge University Press, 1998. ↑5

[KO87] Toshiyuki Katsura and Frans Oort, *Families of supersingular abelian surfaces*, Compositio Mathematica **62** (1987), no. 2, 107–167 (en). ↑126

[Kum15] Abhinav Kumar, *Hilbert modular surfaces for square discriminants and elliptic subfields of genus 2 function fields*, Research in the Mathematical Sciences **2** (2015), no. 1, 24. ↑105

[Lau77] Henry B. Laufer, *On Minimally Elliptic Singularities*, American Journal of Mathematics **99** (1977), no. 6, 1257. ↑12, 43, 45, 48, 50, 64

[Lie13] Christian Liedtke, *Algebraic Surfaces in Positive Characteristic*, Birational geometry, rational curves, and arithmetic, 2013, pp. 229–292. ↑32

[Liu93] Qing Liu, *Courbes stables de genre 2 et leur schéma de modules*, Mathematische Annalen **295** (1993), no. 1, 201–222. ↑101

[LS23] Christopher Lazda and Alexei Skorobogatov, *Reduction of Kummer surfaces modulo 2 in the non-supersingular case*, Épijournal de Géométrie Algébrique **Volume 7** (2023). ↑57, 91, 92

[Mat15] Yuya Matsumoto, *On good reduction of some K3 surfaces related to abelian surfaces*, Tohoku Mathematical Journal **67** (2015), no. 1. ↑91

[Mat23] Yuya Matsumoto, *Supersingular reduction of Kummer surfaces in residue characteristic 2* (2023). ↑91, 98

[Mat24] Yuya Matsumoto, *Inseparable Kummer surfaces* (2024). ↑39

[Mes91] Jean-François Mestre, *Construction de courbes de genre 2 à partir de leurs modules*, Effective methods in algebraic geometry, 1991, pp. 313–334. ↑101

[MI20] Masayoshi Miyanishi and Hiroyuki Ito, *Algebraic Surfaces in Positive Characteristics*, World Scientific, 2020. ↑4

[Mil86] James Milne, *Jacobian Varieties*, Arithmetic geometry, 1986, pp. 167–212. ↑129

[Moo28] Walter L. Moore, *On the Geometry of the Weddle Surface*, The Annals of Mathematics **30** (1928), no. 1/4, 492. ↑81

[Mül10] Jan Steffen Müller, *Explicit Kummer surface formulas for arbitrary characteristic*, LMS Journal of Computation and Mathematics **13** (2010), 47–64. ↑65

[Nag63] Masayoshi Nagata, *Invariants of group in an affine ring*, Kyoto Journal of Mathematics **3** (1963), no. 3. ↑22

[Noe26] Emmy Noether, *Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der Charakteristik p*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1926** (1926), 28–35. ↑25

[Ove21] Otto Overkamp, *Degeneration of Kummer surfaces*, Mathematical Proceedings of the Cambridge Philosophical Society **171** (2021), no. 1, 65–97. ↑91

[Pie22] Andreas Pieper, *Constructing all genus 2 curves with supersingular Jacobian*, Research in Number Theory **8** (2022), no. 2, 32. ↑121

[Pri08] Rachel Pries, *A short guide to p-torsion of abelian varieties in characteristic p*, Computational Arithmetic Geometry **463** (2008), 121–129. ↑119

[Pri67] David Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Mathematical Journal **34** (1967), no. 2. ↑8

[Rei80] Miles Reid, *Canonical 3-folds*, Journées de géometrie algébrique d'angers, 1980, pp. 273–310. ↑23

[Rei87] Miles Reid, *Young's person guide to canonical singularities* (Spencer Bloch, ed.), Proceedings of Symposia in Pure Mathematics, vol. 46.1, American Mathematical Society, Providence, Rhode Island, 1987. ↑7

[RS25] Xavier Roulleau and Alessandra Sarti, *Constructions of Kummer structures on generalized Kummer surfaces*, Kyoto Journal of Mathematics **65** (2025), no. 1. ↑30

[Ryb24] Sergey Rybakov, *Generalized Kummer surfaces over finite fields* (2024). ↑31, 32, 33, 36, 38, 40

[Sch09] Stefan Schröer, *The Hilbert scheme of points for supersingular abelian surfaces*, Arkiv för Matematik **47** (2009), no. 1, 143–181. ↑64, 80

[Sch25] Stefan Schröer, *K3 surfaces over small number fields and Kummer constructions in families* (2025). ↑98

[Shi74] Tetsuji Shioda, *Kummer surfaces in characteristic 2*, Proceedings of the Japan Academy, Series A, Mathematical Sciences **50** (1974), no. 9. ↑41

[SV04] Tanush Shaska and Helmut Völklein, *Elliptic Subfields and Automorphisms of Genus 2 Function Fields*, Algebra, arithmetic and geometry with applications, 2004, pp. 703–723. ↑105

[The25] The LMFDB Collaboration, *The L-functions and modular forms database*, 2025. ↑51 [vdG99] Gerard van der Geer, *Cycles on the Moduli Space of Abelian Varieties*, 1999, pp. 65–89. ↑119, 120, 126

[vL07] Ronald van Luijk, *K3 surfaces with Picard number one and infinitely many rational points*, Algebra & Number Theory **1** (2007), no. 1, 1–15. ↑3

[Vos67] Valentin Evgenievich Voskresenskiĭ, *On two-dimensional algebraic tori II*, Mathematics of the USSR-Izvestiya **1** (1967), no. 3, 691–696. ↑51

[Wag70] Phillip Wagreich, *Elliptic Singularities of Surfaces*, American Journal of Mathematics **92** (1970), no. 2, 419. ↑12, 43, 64

[Wei57] André Weil, *Zum Beweis des Torellischen Satzes*, Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse / 2a: Mathematisch-Physikalisch-Chemische Abteilung, Vandenhoeck & Ruprecht, 1957. ↑13

[WR24] Inc. Wolfram Research, *Mathematica, Version 14.0*, 2024. ↑65