The classification of generalised Kummer surfaces in positive characteristic

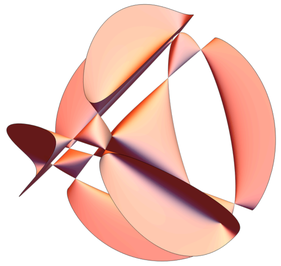
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Question

What is the maximum number of singular points that a quartic surface in $\mathbb{P}^3$ can have over a field of characteristic zero?

Answer: 16 singularities.



These are called Kummer surfaces!

Definition (Kummer surface)

Let A be an abelian surface and let $\iota\colon A \to A$ be the involution sending every point to its inverse with respect to the group law,

$$\iota(P) = -P.$$

The **Kummer surface** of A is the quotient

$$\mathrm{Kum}(A) = A/\langle\iota\rangle.$$

The points fixed by ι are the 2-torsion points $A[2]$. If the characteristic of the base field is not two, there are sixteen of them, so $\mathrm{Kum}(A)$ has sixteen singularities of type A_1, and the desingularisation of $\mathrm{Kum}(A)$ is a K3 surface.

Kummer surfaces are one of the most important examples of K3 surfaces because their geometry can be easily understood from the one of abelian surfaces.

How can we generalise them?

Definition (Action preserving the group law)

Let G be a finite group acting on an abelian surface A. We say that G **preserves the group law** if the elements of G act as endomorphisms of A, so that there is a monomorphism

$$G \hookrightarrow \mathrm{End}(A)^\times.$$

Definition (Generalised Kummer surface)

Let A be an abelian surface and G a finite group acting on A. We say that the quotient A/G is a **generalised Kummer surface** if the minimal resolution of singularities of A/G is a K3 surface.

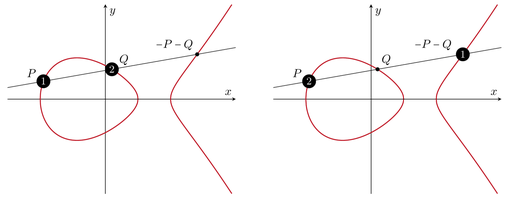
Non-example: If G is generated by a translation by a torsion point of A, then A/G is again an abelian surface, so it is never a generalised Kummer surface.

Example (Action of order three)

Let E be an elliptic curve and $A = E \times E$. We define an automorphism of A of order three, preserving the group law, by

$$\tau_3(P, Q) = (-P - Q,\ P).$$

For every two points $P, Q \in E$, the line through them meets E in a third point $-P-Q$, and τ_3 permutes cyclically the pairs of points on this line:



The points fixed by τ_3 are the points (P, P) with $P \in E[3]$. In characteristic different from three, $(E \times E)/\langle\tau_3\rangle$ has nine A_2 singularities, and it is a generalised Kummer surface.

Theorem (Katsura [Kat87])

Let A be an abelian surface over an algebraically closed field of characteristic zero, and let G be a finite group acting on A and preserving the group law. If A/G is a generalised Kummer surface, then G is one of the groups below, and the singularities of A/G are as follows. Conversely, every case occurs.

G	Singularities of A/G
C_2	$16A_1$
C_3	$9A_2$
C_4	$4A_3 + 6A_1$
C_6	$A_5 + 4A_2 + 5A_1$
Q_8	$2D_4 + 3A_3 + 2A_1$
Q_8	$4D_4 + 3A_1$
Q_{12}	$D_5 + 3A_3 + 2A_2 + A_1$
$\mathrm{SL}_2(\mathbb{F}_3)$	$E_6 + D_4 + 4A_2 + A_1$

Problems in positive characteristic

When we try to extend this theory to characteristic p, we encounter some problems when p divides $|G|$.

For instance, in the case of Kummer surfaces in characteristic two, if A is **supersingular** (meaning in this case that the 2-torsion is trivial), the desingularisation of $\mathrm{Kum}(A)$ is not a K3 surface.

The ***p*-rank** $f(A) \in \{0, 1, 2\}$ of A is defined by $\#A[p] = p^{f(A)}$, and A is supersingular if and only if $f(A) = 0$.

The number and type of singular points depend on the 2-rank:

- If $f(A) = 2$, then $\mathrm{Kum}(A)$ has singularities $4D_4^1$.
- If $f(A) = 1$, then $\mathrm{Kum}(A)$ has singularities $2D_8^2$.
- If $f(A) = 0$, then $\mathrm{Kum}(A)$ has one elliptic singularity, and it is not a K3 surface.

What are the possible generalised Kummer surfaces in positive characteristic?

Theorem (Rybakov [Ryb24])

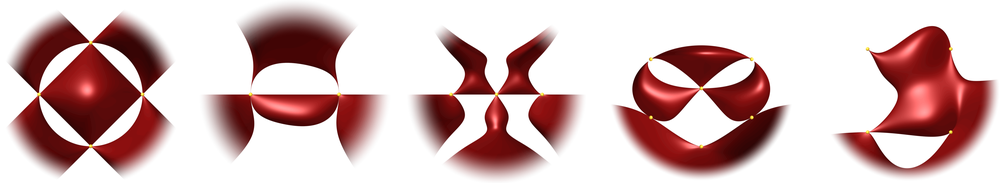
If p does not divide $|G|$, then, besides the groups in Katsura's theorem, the following groups are also possible, and in all these cases A must be supersingular:

- C_5, C_{10}, Q_{20}, $\mathrm{SL}_2(\mathbb{F}_5)$, if $p \equiv \pm 2 \bmod 5$;
- C_8, Q_{16}, $\mathrm{ESL}_2(\mathbb{F}_3)$, if $p \equiv \pm 3 \bmod 8$;
- C_{12}, Q_{24}, if $p \equiv \pm 5 \bmod 12$.

Theorem (Gonzalez-Hernandez [GH26])

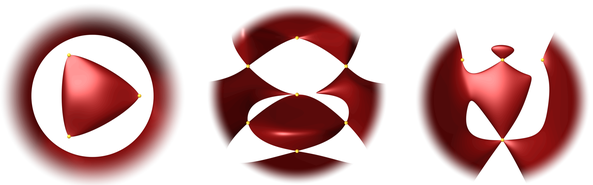
Let A be an abelian surface over an algebraically closed field of characteristic p, and let G be a finite group acting on A and preserving the group law, with p dividing $|G|$. Then A/G is a generalised Kummer surface if and only if G, p and A are as in one of the following eight cases. In particular, A is never supersingular.

$p = 2$



G	C_2	C_2	C_4	C_6	Q_{12}
p-rank $f(A)$	2	1	2	2	2
Singularities of A/G	$4D_4^1$	$2D_8^2$	$2E_7^3 + D_4^1$	$E_6^1 + D_4^1 + 4A_2$	$E_8^4 + E_7^3 + 2A_2$

$p = 3$



G	C_3	C_6	$\mathrm{SL}_2(\mathbb{F}_3)$
p-rank $f(A)$	2	2	2
Singularities of A/G	$3E_6^1$	$E_7^1 + E_6^1 + 5A_1$	$E_8^2 + E_6^1 + D_4 + A_1$

Furthermore, all of these surfaces can be constructed explicitly as quotients of products of elliptic curves.

Key ideas of the proofs of the theorems

1. Understanding the local fundamental groups of ADE singularities in positive characteristic [Art77].
   → *This determines which groups G can act on A.*
2. Studying the possible actions of G on the rational Tate module $V_\ell(A)$.
   → *This determines the number of singular points of A/G and their possible types.*
3. Analysing $H^2(X, \mathbb{Q}_\ell)$, where X is the minimal resolution of A/G, in terms of the action of G on A and the exceptional divisors of the singularities.
   → *This allows us to rule out some cases.*

Want more detail?
Read the source!

References

[Art77] M. Artin, *Coverings of the rational double points in characteristic p*, Complex Analysis and Algebraic Geometry, Cambridge Univ. Press (1977), 11–22.

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