Singular fibres of elliptic surfaces

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1 Algebraic surfaces

1.1 Divisors of algebraic surfaces

In this section, we will work with irreducible smooth projective surfaces X over an algebraically closed field k. In these surfaces, we can consider the **divisor group** consisting of formal sums of irreducible subvarieties of codimension one, i.e. curves on X:

$$\mathrm{Div}(X) = \Big\{ \sum_{i=1}^{m} n_i C_i \colon m \in \mathbb{N}, n_i \in \mathbb{Z}, C_i \subset X \text{ irreducible curve} \Big\}.$$

A divisor is called **effective** if all $n_i \geq 0$ (assuming the curves C_i to be all distinct). The **degree** of a divisor is defined through the degrees of the curves C_i by the formula

$$\deg(X) = \sum_{i=1}^{m} n_i \deg(C_i).$$

As it happens in the case of curves, a non-zero rational function $f \in k(X)^\times$ gives rise to a divisor $(f) \in \mathrm{Div}(X)$ of degree zero defined as the difference of the zero divisor and the pole divisor of f. Such kind of divisors are called **principal divisors**.

Two divisors D and D' are said to be **linearly equivalent** ($D \sim D'$) if and only if $D - D' = (f)$ for some $f \in k(X)$. The quotient of the divisor group $\mathrm{Div}(X)$ by linear equivalence is called the **Picard group** of X, $\mathrm{Pic}(X)$, and the equivalence classes are known as **divisor classes**. As seen in Hartshorne [Har77, II, Chapter 6], there is a bijection that associates a divisor class D with the invertible sheaf $\mathcal{O}_X(D)$ consisting of functions in $k(X)$ which are regular outside of D and may have zeroes of prescribed multiplicities if D is not effective. Then, the Picard group can be understood as the group of invertible sheaves up to isomorphism with the structure sheaf $\mathcal{O}_X$ as identity, tensor product as multiplication and the dual invertible sheaf as inverse.

An alternative, more cohomological, approach to the Picard group is the following. If we denote by $\mathcal{O}_X^*$ the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation, it is an easy exercise [Har77, III, Exercise 4.5] to prove that

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*).$$

The equivalence between D and $\mathcal{O}_X(D)$ allows us to extract information about a divisor from the sheaf cohomology of $\mathcal{O}_X(D)$. In that regard, the following definition is fundamental:

Definition 1.1. *The **canonical sheaf** of an n-dimensional variety X is the highest exterior power of the sheaf of differentials Ω_X^1*

$$\omega_X = \bigwedge^n \Omega_X^1.$$

The identification of divisors and invertible sheaves associates to ω_X the **canonical divisor** K_X. More explicitly, we can compute this divisor through the following strategy:

Let $x_1, \dots, x_n$ be a system of local coordinates of our variety, we can consider a volume element $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. Then, if we pick n elements $f_1, \dots, f_n \in K(X)$ that form a (separable) transcendence basis, that is, such that $k(f_1, \dots, f_n) \subset k(X)$ is a finite algebraic extension; and $0 \neq g \in k(X)$, we get that $s = g\, df_1 \wedge \cdots \wedge df_n$ is a rational n-form. Locally, we can compare it to the local volume element $dx_1 \wedge \cdots \wedge dx_n$ by means of the Jacobian determinant J:

$$s = g\, df_1 \wedge \cdots \wedge df_n = J \cdot g\, dx_1 \wedge \cdots \wedge dx_n, \text{ where } J = J\Big(\frac{f_1, \dots, f_n}{x_1, \dots, x_n}\Big) = \det\Big(\frac{\partial f_i}{\partial x_j}\Big).$$

By the chain rule, a different choice of local coordinates $x_1, \dots, x_n$ multiplies J by an invertible function, so that the zeroes and poles of J are well defined. This makes it possible to determine the zeroes and poles of s, by defining the valuation of s at a prime divisor Γ by

$$v_\Gamma(s) := v_\Gamma(J \cdot g).$$

We can then set

$$\operatorname{div}(s) = \sum_\Gamma v_\Gamma(s)\, \Gamma.$$

The canonical class of X can be defined as $K_X = \operatorname{div}(s)$ for any rational n-form. This is a well defined divisor class, because two different n-forms s and s' are related by $s = hs'$ with $0 \neq h \in k(X)$ a rational function, and then, $\operatorname{div}(s) = \operatorname{div}(s') + \operatorname{div}(h)$.

Example 1.1. Let X be a smooth surface of degree d in $\mathbb{P}^3$ and let H be a hyperplane section. Then, $K_X \sim (d-4)H$ and $\omega_X = \mathcal{O}_X(d-4)$.

In order to compute the canonical class of divisors, we can also use the following practical result [Rei962, Section A.11].

Theorem 1.1 (Adjunction formula). *Suppose that Y is a smooth divisor on X. Then,*

$$K_Y = (K_X + Y)\mid_Y .$$

Here, the restriction of divisor classes means to first take a divisor D on Y with $D \sim K_X + Y$ and such that D does not contain Y and then, intersect D with Y to get the divisor class $(K_X + Y)\mid_Y$.

Once we know the canonical divisor, we can define the **geometric genus** of X as $p_g(X) = h^0(X, \omega_X)$. This will be an important invariant of surfaces, as we will see in section 1.5.

Example 1.2. Let X be a smooth surface of degree d in $\mathbb{P}^3$. Then,

$$p_g(X) = \binom{d-1}{3}.$$

1.2 Algebraic equivalence

Linear equivalence is not the only equivalence relation that can be defined on divisors of a surface.

Definition 1.2. *A divisor D on X is said to be* ***algebraically equivalent*** *to zero (denoted $D \approx 0$) if there is a connected scheme T and an effective divisor $\overline{D}$ on $X \times T$, such that $\overline{D}$ is flat over T and there exist two points $t_1, t_2 \in T$ whose fibres $\overline{D}_{t_1}, \overline{D}_{t_2}$ on $\overline{D}$ satisfy*

$$D = \overline{D}_{t_2} - \overline{D}_{t_1}.$$

We say that two divisors D_1, D_2 are ***algebraically equivalent*** *if $D_1 - D_2 \approx 0$.*

Intuitively, the idea behind this definition is that two divisors are equivalent if there is an algebraic family D_t with $t \in T$ connecting D_1 and D_2.

A key observation is that by picking $T = \mathbb{P}^1$, it is easy to check that linear equivalence implies algebraic equivalence. This fact suggests that by considering the quotient of the divisor group by algebraic equivalence, we will get a group that is related to the Picard group. That is indeed the case.

Definition 1.3. *The* ***Néron-Severi group*** *of X is the divisor group $\text{Div}(X)$ modulo algebraic equivalence*

$$\text{NS}(X) = \text{Div}(X)/\approx.$$

As a consequence of the theorem of the base (valid for smooth projective varieties), we have the following result:

Theorem 1.2. *The Néron-Severi group of a smooth projective variety is a finitely generated abelian group* [GH94, Section 3.5].

The rank of the Néron-Severi group of X is called the **Picard number** and denoted by $\rho(X)$. This quantity will be the main object of study of this project.

Another alternative view of the Néron-Severi group is the following: if we assume X to be a complex manifold, GAGA-style [Ser56] results allow us to change between the algebraic and the analytic categories (in such a way such that, for instance $\mathcal{O}_X$ can be identified with the sheaf of holomorphic functions on X and $\mathcal{O}_X^*$ with the subsheaf consisting of the non-vanishing holomorphic functions). The exponential function then gives a sheaf homomorphism $\exp\colon \mathcal{O}_X \to \mathcal{O}_X^*$ which allows us to construct an exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0.$$

At the same time, this sequence induces a long cohomology sequence

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \cdots$$

where the connection morphism c_1 is known as the **first Chern map**. Then, the image of c_1 can be identified with the Néron–Severi group and we naturally get that $\mathrm{NS}(X)$ is a finitely generated abelian group from the fact that it is a subgroup of $H^2(X,\mathbb{Z})$. The kernel of c_1, on the other hand, is isomorphic to the quotient

$$\mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}),$$

which is known as the **Picard variety** of X. This variety is an abelian variety, and if instead of working with X a surface, we consider a curve C; $\mathrm{Pic}^0(C)$ is isomorphic to the Jacobian variety of C.

Going back to the surface case, we have that the Picard group fits in the following exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

For fields of positive characteristic, this theory does not hold as such, but we have a similar notion by using instead ℓ-adic étale cohomology, as $\mathrm{NS}(X)$ has a map to $H^2_{\text{ét}}(X,\mathbb{Q}_\ell)$. There are some differences with respect of the characteristic zero case; to give an example of how these theories are different, Igusa constructed an example of a smooth projective surface X with $\mathrm{Pic}^0(X)$ non-reduced, and hence not an abelian variety [Igu5511]. But this theory is still very useful to deduce information about the characteristic zero case, as we will see in section 7.3.

1.3 Intersection theory

One of the reasons why it is so interesting to study the Picard group of a smooth algebraic surface X is because we can endowed it with an intersection form [Sha13, Chapter IV].

Theorem 1.3 (Intersection pairing). *Let X be a smooth projective surface. Then, there exists a pairing*

$$\begin{aligned} \mathrm{Div}(X) \times \mathrm{Div}(X) &\longrightarrow \mathbb{Z} \\ (D_1,\, D_2) &\longmapsto D_1 \cdot D_2 \end{aligned}$$

satisfying the following properties:

1. *$D_1 \cdot D_2$ is a bilinear and symmetric pairing.*
2. *$D_1 \cdot D_2$ depends on D_1 and D_2 only up to linear equivalence, i.e.*

$$D_1 \sim D_3 \quad \Longrightarrow \quad D_1 \cdot D_2 = D_3 \cdot D_2.$$

3. *If C and D are smooth curves that do not have any common components,*

$$C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P$$

where $(C \cdot D)_P = \dim_k \mathcal{O}_{Y,P}/(f,g)$ and f and g are local equations for C and D respectively at the point P.

The second condition implies that this intersection pairing extends to $\mathrm{Pic}(X)$.

Example 1.3. In $\mathbb{P}^2$, if we let $C : X^4 = Y^3Z$ and $D : X^2 = YZ$, we get

$$\begin{aligned}(C \cdot D)_{[0:0:1]} &= \dim_k \frac{k[X,Y,Z]_{[0:0:1]}}{(X^4 - Y^3Z, X^2 - YZ)} = \dim_k \frac{k[x,y]_{(0,0)}}{(x^4 - y^3, x^2 - y)} \\ &= \dim_k \frac{k[x]_{(0)}}{((x^2-1)x^4)} = \dim_k \frac{k[x]_{(0)}}{(x^4)} \\ &= \dim_k(k \oplus kx \oplus kx^2 \oplus kx^3) = 4.\end{aligned}$$

Similarly $(C \cdot D)_{[1:1:1]} = 1$, $(C \cdot D)_{[-1:1:1]} = 1$, and $(C \cdot D)_{[0:1:0]} = 2$. Therefore $C \cdot D = 8$.

More generally, on $\mathbb{P}^2$, the class of a divisor is given by its degree. In $\mathbb{P}^2$ we also have the following result:

Theorem 1.4 (Bezout's). *If two plane algebraic curves C and D of degrees n_1 and n_2 have no component in common, they have $n_1 n_2$ intersection points, counted with their multiplicity.*

We therefore deduce that the intersection pairing in $\mathbb{P}^2$ is given by

$$\begin{aligned}\mathrm{Div}(\mathbb{P}^2) \times \mathrm{Div}(\mathbb{P}^2) &\longrightarrow \mathbb{Z} \\ (D_1,\ D_2) &\longmapsto \deg(D_1)\deg(D_2)\end{aligned}$$

and therefore, this pairing over $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ corresponds to multiplication, as we have seen in the explicit example.

Assuming the divisors have no common components, it is easy to calculate the intersection number through computing the multiplicity of intersection at all points of intersection. However, this method does not work when the divisors have common components.

An option to bypass this difficulty is to find a linearly equivalent divisor to one of our divisors so that it meets transversely with the other. This strategy can be used to compute the self-intersection of curves on a surface. For instance, it can be proved in this way that the self-intersection number of a line L on a non-singular surface X of degree d in $\mathbb{P}^3$ is $L \cdot L = L^2 = 2 - d$ [Sha13, Example 4.7] and that the self-intersection of a hyperplane section is $H^2 = d$, as we will later see.

There is another, more elegant solution, to the problem of computing intersection numbers involving cohomology.

Definition 1.4. *Let $\mathcal{F}$ be an invertible sheaf on X. Then, the* ***Euler characteristic*** *of $\mathcal{F}$ is defined as*

$$\chi(\mathcal{F}) = h^0(X,\mathcal{F}) - h^1(X,\mathcal{F}) + h^2(X,\mathcal{F}).$$

For any two divisors $D_1, D_2 \in \mathrm{Div}(X)$, we can then define their **intersection number** as

$$D_1 \cdot D_2 = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D_1)) - \chi(\mathcal{O}_X(-D_2)) + \chi(\mathcal{O}_X(-D_1 - D_2)).$$

It can be proven that this definition matches all the conditions of theorem 1.3, with the additional advantage that it is well defined for divisors with common components. As a matter of fact, it can be deduced that for a divisor D associated to a non-singular curve,

$$D^2 = \deg_D(\mathcal{O}_D(D)) = \deg_D(\mathcal{N}_{X|D}),$$

where $\mathcal{N}_{X|D}$ is the normal sheaf to X along D [Har77, V, Example 1.4.2].

Using the intersection pairing, we can restate both the Riemann-Roch theorem and the adjunction formula for curves on surfaces [Har77, V, Propositions 1.5 & 1.6]:

Theorem 1.5 (Riemann-Roch). *Let D be a divisor on a smooth surface X. Then,*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \tfrac{1}{2}\big(D \cdot (D - K_X)\big).$$

Theorem 1.6 (Adjunction formula). *If C is a nonsingular curve on the surface X that has genus $g(C) = 1 - \chi(\mathcal{O}_C)$, then*

$$2g(C) - 2 = (K_X + C) \cdot C = K_X \cdot C + C^2.$$

The intersection pairing also allows us to define an equivalence relation in the divisor class group of a surface.

Definition 1.5. *We say that two divisors D_1 and D_2 of X are* ***numerically equivalent*** *($D_1 \equiv D_2$) if $D_1 \cdot C = D_2 \cdot C$ for every $C \in \operatorname{Div}(X)$.*

Let $\operatorname{Num}(X)$ be the quotient of the divisor group by the numerical equivalence relation

$$\operatorname{Num}(X) = \operatorname{Div}(X) \,/\, \equiv .$$

It can be proven that algebraic equivalence implies numerical equivalence. As a matter of fact, we have that

$$\operatorname{Num}(X) = \operatorname{NS}(X) \,/\, \operatorname{NS}(X)_{\text{tors}}.$$

Hence we obtain an induced pairing on $\operatorname{NS}(X)$. Cohomologically, this pairing commutes with the first Chern map in the following sense: $H^2(X, \mathbb{Z})$ and $H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ come equipped with a cup-product $\cup$, a symmetric pairing whose image sits inside $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ or, respectively, $H^4_{\text{ét}}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$. For divisor classes D_1, D_2, one gets

$$D_1 \cdot D_2 = c_1(D_1) \cup c_1(D_2).$$

Finally, it is good to mention that there are several instances where some of these notions of equivalence of divisors that we have seen coincide. For instance, on a smooth hypersurface $X \subset \mathbb{P}^3$ all three linear, algebraic and numerical equivalences are the same and we attain complete control over the curves on X by computing intersection numbers.

1.4 Lattice theory

Many of the most important tools that we will use to analyse surfaces come from the fact that $\mathrm{Num}(X)$ forms a **lattice**, a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear pairing (given in this case by the intersection pairing). For that reason, we will go through some concepts related to lattices that are important to study [CS99] in order to better understand the Néron-Severi group of a variety.

Definition 1.6. *Let $\langle\ ,\ \rangle$ denote the bilinear pair of a lattice L. Then, the **Gram matrix** of L with respect to a given basis $\{x_1, \dots, x_n\}$ is the symmetric matrix given by*

$$M_L = (\langle x_i, x_j \rangle)_{ij}$$

The **determinant** or **discriminant** of L is the determinant of its Gram matrix . A lattice is called **unimodular** if $|\det L| = 1$.

An **integral lattice** is a lattice with a $\mathbb{Z}$-valued pairing. An integral lattice is called **even** if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$ and **odd** otherwise.

A lattice is called **positive-definite** if for every non-zero element $x \in L$, $\langle x, x \rangle > 0$. Likewise, it is negative-definite if $\langle x, x \rangle < 0$. It can easily be proven that a lattice is positive-definitive if and only if its Gram matrix is a positive-definite matrix, and the same is true for negative-definite lattices. We will use the notation L^- to refer to the lattice whose Gram matrix is the opposite of the lattice L.

The **signature** of L is defined as the signature of its Gram matrix M_L, i.e. the pair (s_+, s_-) where s_+ is the number of positive eigenvalues of M_L and s_- is the number of negative eigenvalues.

There are some particular types of lattices of special relevance, which are known as **root lattices.** Let L be a definite even integral lattice. A **root** of L is an element $x \in L$ with $\langle x, x \rangle = \pm 2$. We will denote by $\mathcal{R}(L)$ the set of roots of L. Then,

Definition 1.7. *A definite even integral lattice is called a **root lattice** if its generated by its roots.*

An important result in Lie algebras states that any positive-definite even integral root lattice is isometric to an orthogonal sum of root lattices of three types: A_k with $k \geq 1$, D_m with $m \geq 4$ and E_n with $n \in \{6, 7, 8\}$. These lattices can be defined in the following way:

Definition 1.8. *A lattice L of rank r is a root lattice of type A_r, D_r or E_r if there exists a basis $\{\alpha_1, \dots, \alpha_r\} \subset \mathcal{R}(L)$ of L such that the following holds: for $1 \leq i < j \leq r$, $\langle \alpha_i, \alpha_j \rangle = 0$ unless*

$$\begin{array}{llllll} A_n & \langle \alpha_i, \alpha_j \rangle = -1 & \iff & i+1=j. & & \\ D_n & \langle \alpha_i, \alpha_j \rangle = -1 & \iff & i+1=j<n, & \textit{or} & i=n-2, \quad j=n. \\ E_n & \langle \alpha_i, \alpha_j \rangle = -1 & \iff & i+1=j<n, & \textit{or} & i=3, \quad j=n. \end{array}$$

1.5 Some invariants of surfaces

We will now present some of the main invariants that we can use to analyse an algebraic surface X. Let $H^i(X)$ represent the i-th singular cohomology group over $\mathbb{C}$ in the case of characteristic zero or the i-th ℓ-adic étale cohomology group for some prime ℓ different than the characteristic of definition, for positive characteristic.

Then, we define the i-th **Betti number** by

$$b_i(X) = \dim H^i(X).$$

Using these Betti numbers, the **topological Euler number** is defined as

$$e(X) = \sum_{i=0}^{4} (-1)^i b_i(X).$$

It is important not to confuse this invariant with the **Euler characteristic** of X, which is defined as $\chi(X) = \chi(\mathcal{O}_X)$.

The **arithmetic genus** is defined as

$$p_a(X) = (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1).$$

The **irregularity** of X is defined as

$$q(X) = h^1(X, \mathcal{O}_X).$$

In the case of curves, the irregularity, the geometric and arithmetic genus are all equal to the genus of the curve. However, in surfaces in characteristic zero, the irregularity is equal to the dimension of the Picard variety, which also corresponds to the difference between the values of the geometric and arithmetic genus. Furthermore, all the Euler characteristic, the geometric genus and the irregularity of a smooth projective surface are invariant by birational morphisms between smooth surfaces.

In characteristic zero, we can also use Hodge theory to further decompose the complex cohomology according to the sheaf of regular p-forms Ω^p_X on the surface. By setting $H^{p,q}(X) = H^q(\Omega^p_X)$, we obtain the decomposition ([BHPV04, IV, Theorem 2.9])

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

We will define the **Hodge numbers** of the surface X by

$$h^{p,q}(X) = \dim H^{p,q}(X, \mathbb{C}).$$

These numbers are classically represented in a diagram known as the **Hodge diamond**:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

The fact that $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and Serre duality, also allow us to deduce the following relations among the Hodge numbers:

$$h^{p,q}(X) = h^{q,p}(X), \qquad h^{p,q}(X) = h^{2-p,\,2-q}(X).$$

Combining all that we know allows us to deduce that the Hodge diamond is given by

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

Another key relation between these invariants comes from **Noether's formula**

$$12\,\chi(X) = e(X) + K_X^2.$$

As previously mentioned, regardless of the characteristic, there is a map from $\mathrm{NS}(X)$ to $H^2(X)$. It has been proven that this induces a bound for the Picard number

$$\rho(X) \leq b_2(X).$$

In characteristic zero, there is even a stronger bound given by

$$\rho(X) \leq h^{1,1}(X) = b_2(X) - 2h^{2,0}(X).$$

This is a direct consequence of Lefschetz theorem on $(1,1)$-classes [BHPV04, IV, Theorem 2.12], which states that given a surface with $h^{2,0}(X) > 0$, we have that

$$\mathrm{NS}(X) = H^{1,1}(X) \cap i(H^2(X,\mathbb{Z})),$$

where i is the morphism to $H^2(X,\mathbb{C})$ induced by tensoring $H^2(X,\mathbb{Z})$ by $\mathbb{C}$.

2 Singularities and birational transformations

So far, we have been working with smooth surfaces. While working with singular surfaces is slightly trickier, there are still multiple techniques through which we can relate these with birationally equivalent smooth surfaces. One of these is the use of blow-ups.

2.1 Blow-up of surfaces

In this project, we will only consider the blow-up of points in surfaces, but other subvarieties can be blown-up as well. For a review of this theory, one can check Shafarevich's book [Sha13, Section 2.4].

Let X be a surface, and let P a point in X. Furthermore, let U be an open neighborhood of P in X, and let (x_1, x_2) be local coordinates, so that the P corresponds to $(0, 0)$. We can then consider

$$\tilde{U} = \{((x_1, x_2), [y_1 : y_2]) \in U \times \mathbb{P}^1 \mid x_1 y_2 = x_2 y_1\}.$$

Clearly, we have a map $\sigma_U : \tilde{U} \to U$, which is the projection onto the first factor. We then have that if $(x_1, x_2) \neq (0, 0)$, then $\sigma_U^{-1}((x_1, x_2)) = \{((x_1, x_2), [x_1 : x_2])\}$, that $\sigma_U^{-1}(P) = \{P\} \times \mathbb{P}^1$, and therefore σ_U defines an isomorphism between $\sigma_U^{-1}(U \setminus \{P\})$ and $U \setminus \{P\}$. If we now take the gluing of X and $\tilde{U}$ along $X \setminus \{P\}$ and $U \setminus \{P\}$, we obtain a surface Y together with a morphism $\sigma : Y \to X$ such that

1. $\sigma^{-1}(P)$ is a smooth rational curve that is contracted to P through σ.
2. When restricted to the open subsets $Y \setminus \sigma^{-1}(P)$ and $X \setminus \{P\}$, σ gives an isomorphism.

The morphism σ is known as the **blow-up** of X along P. The curve $\sigma^{-1}(P)$ is called the **exceptional divisor** and is usually denoted by E.

Let C be a curve in X. The **strict transform** of C under the blow-up of X along P is the closure $\tilde{C}$ in Y of $\sigma^{-1}(C \setminus \{P\})$. The strict transform of a curve is always a curve. If $P \notin C$, then $\tilde{C} = \sigma^{-1}(C)$, so that $\tilde{C} = \sigma^* C$, the pull-back of the divisor class of C through σ. For the more general case where C is an irreducible curve passing through P with multiplicity m, we have [Har77, V, Proposition 3.6.] that

$$\sigma^* C = \tilde{C} + mE$$

In the case that X is a smooth surface and $\sigma : Y \to X$ the blow-up along P, we can say even more:

- For every two divisors D_1, D_2 on X we have that

$$\sigma^* D_1 \cdot \sigma^* D_2 = D_1 \cdot D_2 \qquad\qquad \sigma^* D_1 \cdot E = \sigma^* D_2 \cdot E = 0$$

- The canonical divisor transforms via pull-back of divisor classes through σ^* asW $K_Y = \sigma^* K_X + E$.

- The blow-up σ induces an isomorphism

$$\begin{aligned} \mathrm{Pic}(X) \oplus \mathbb{Z} &\longrightarrow \mathrm{Pic}(Y) \\ (\mathcal{O}_X(D), n) &\longmapsto \mathcal{O}_Y(\sigma^* D + nE) \end{aligned}$$

  which extends to the Néron-Severi groups. In particular $\rho(Y) = \rho(X) + 1$ and $h^{1,1}(Y) = h^{1,1}(X) + 1$.

- The exceptional divisor satisfies that $E^2 = -1$ on Y [Sha13, Section 2.4.4].

Generally, any smooth rational curve E on X self-intersection -1 is called a (-1)-**curve**. These curves play an important role in the study of surfaces up to birational equivalence, due to the following results:

Theorem 2.1. *Let $f : X \to Y$ be a birational morphism between two smooth projective surfaces. Hence f is the composition of a finite number of blow-ups and isomorphisms.*

The following theorem states what blow-ups do to surfaces:

Theorem 2.2 (Castelnuovo contractibility criterion). *Let Y be a smooth projective surface, and suppose it has a (-1)-curve E. Then there is a smooth projective surface X and a point $P \in X$ such that there is a blow-up $\sigma : Y \to X$ with $\sigma^{-1}(P) = E$.*

This theorem basically states that whenever on a smooth surface X there exists a (-1)-curve, we can contract it to get a smooth projective surface whose Picard number is smaller. This gives us a sense in which we can speak of minimality.

Definition 2.1. *A smooth surface X is called* ***minimal*** *if every birational morphism $f : X \to Y$ is an isomorphism. A* ***minimal model*** *for a surface X is a surface X_0 which is minimal and birational to X.*

As a corollary of the Castelnuovo criterion we have the following result:

Corollary 2.1. *A smooth surface X is minimal if and only if it does not contain any (-1)-curve. In particular, every smooth surface has a minimal model.*

It is important to mention, though, that minimal models of surfaces need not be unique, for instance, it can be checked that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are both minimal models of the blow-up of $\mathbb{P}^2$ at two points [Har92, Example 7.22].

The importance of the existence of minimal models is that it reduces the problem of studying smooth surfaces up to birational equivalence, to studying their minimal models up to isomorphism.

2.2 Du Val singularities

Besides helping us to study smooth surfaces, blow-ups play a key rule in the process of **desingularisation of surfaces**, that is, finding a birational morphism from a nonsingular surface to a given singular surface. As a matter of fact, in many set ups, blow-ups are all you need: Hironaka proved in 1964 that in characteristic zero, any singularity can be resolved by a succession of blow-ups [Hir64], and in 2009, Cossart and Piltant extended these results in positive characteristic for varieties up to dimension 3 [CP094].

The study of singularities is a big area of research on itself, so in this piece of work, we will limit ourselves to only working with surfaces with, at worst, **canonical singularities**.

Definition 2.2. *A point $P \in X$ of a normal surface is called a canonical, **du Val singularity** if there exists a minimal resolution $f : Y \to X$ contracting curves $C_1, \dots, C_r$ to P, such that $K_Y \cdot C_i = 0$ for all i, where K_Y is the canonical class of Y.*

The idea behind studying surfaces that have this kind of singularities, besides being able of desingularise them with blow-ups, is that they are well-behaved [Rei87]. For instance, let X_0 be a smooth surface and degenerate it to a surface X with isolated du Val singularities. Then, the minimal resolution Y of X lies in the same deformation class as X_0. In particular, the minimal resolution Y shares several invariants with the original surface X_0 (Betti numbers, Euler number and characteristic, and Hodge numbers over $\mathbb{C}$ in the characteristic zero case).

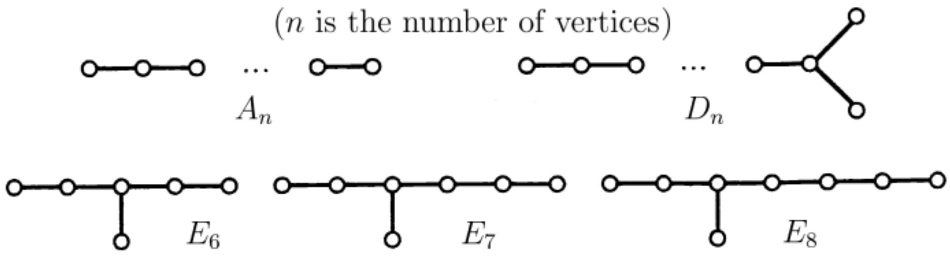
Over algebraically closed fields in characteristic zero, it is always possible to find changes of coordinates such that any du Val singularity can be deformed to a hypersurface singularity given by one of the following equations:

$$\begin{aligned}
A_n &: \quad x^2 + y^2 + z^{n+1} = 0 && \text{for } n \geq 1,\\
D_n &: \quad x^2 + y^2 z + z^{n-1} = 0 && \text{for } n \geq 4,\\
E_6 &: \quad x^2 + y^3 + z^4 = 0, &&\\
E_7 &: \quad x^2 + y^3 + yz^3 = 0, &&\\
E_8 &: \quad x^2 + y^3 + z^5 = 0 &&
\end{aligned}$$

In positive characteristic the same is true, with the small difference that in characteristics $2, 3$ and 5 there are additional possibilities, as proven by Artin [Art755].

The denomination of these singularities matches the one we used to describe root lattices and, indeed, there is a connection between lattices, singularities and simple Lie groups, known as the **McKay correspondence**. If $P \in X$ is a du Val singularity, the bunch of curves $C_1, \dots, C_r$ contracted to P under a resolution can be drawn as a graph: each curve C_i is represented by a node, and intersecting curves C_i and C_j are joined by an edge.

These singularities can then be identified with the following simply laced Dynkin diagrams:



which will have connections with the Gram matrices (A_n^-, D_n^-, E_n^-) of the root lattices as we will see in section 3.2.

Finally, there is a third way of characterising du Val singularities in characteristic zero and that is as quotients of the affine plane by finite subgroups $\Gamma \subset SL(2, \mathbb{C})$. The idea behind this is that Γ acts linearly on $\mathbb{A}^2_{\mathbb{C}}$ by matrices with trivial determinant, and the singularities are obtained as the quotient space $\mathbb{A}^2/\Gamma$ that we can compute from the Γ-invariant monomials in $\mathbb{A}^2$ [Rei87].

The groups associated to each du Val singularity are the cyclic group of order $n+1$ for A_n, the binary dihedral group of order $4(n-2)$ for D_n, and the binary tetrahedral, binary octahedral and binary icosahedral groups for the E_6, E_7 and E_8 singularities respectively.

2.3 Some examples of GIT quotients

This last characterisation of du Val singularities consists in understanding them as GIT (geometric invariant theory) quotients, by studying the way that a group scheme acts on an affine scheme. In this section, we would like to present a few more examples of GIT quotients that will be relevant later on.

Definition 2.3. *Let $(a_1, \ldots, a_n)$ with $a_i \in \mathbb{N}$, and define an action of $\mathbb{G}_m$ on $\mathbb{A}^n \setminus \{0\}$ by*

$$\begin{aligned} \mathbb{G}_m \times \mathbb{A}^n \setminus \{0\} &\longrightarrow \mathbb{A}^n \setminus \{0\} \\ (\lambda, (x_1, \cdots, x_n)) &\longmapsto (\lambda^{a_0} x_0, \cdots, \lambda^{a_n} x_n). \end{aligned}$$

*We define the **weighted projective space** $\mathbb{P}(a_1, \cdots, a_n)$ as the quotient of $\mathbb{A}^n \setminus \{0\}$ by the previous action.*

Usually, when we work with weighted projective spaces we work with reduced models given by well-formed weights:

Definition 2.4. *We say that a weight $(a_1, \ldots, a_n)$ is **well-formed** if any $n-1$ of the a_i are coprime. That is, for all $1 \leq i \leq n$*

$$\gcd(a_1, \ldots, \widehat{a_i}, \ldots, a_n) = 1$$

The second example is a generalisation of weighted projective spaces for products of the projective space.

Definition 2.5. *Let*

$$M = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1m} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & \cdots & b_{2m} \end{array}\right) \in \mathrm{Mat}_{2\times(n+m)}(\mathbb{Z})$$

*We define the **scroll** $\mathbb{F}_M$ as the quotient of $(\mathbb{A}^n \setminus \{0\}) \times (\mathbb{A}^m \setminus \{0\})$ by the following action of $\mathbb{G}_m \times \mathbb{G}_m$:*

$$\begin{aligned} (\mathbb{G}_m \times \mathbb{G}_m) \ \times \ ((\mathbb{A}^n \setminus \{0\}) \times (\mathbb{A}^m \setminus \{0\})) &\longrightarrow (\mathbb{A}^n \setminus \{0\}) \times (\mathbb{A}^m \setminus \{0\}) \\ ((\lambda, 1), (t_1, \ldots, t_n; x_1, \ldots, x_m)) &\longmapsto (\lambda^{a_{11}} t_1, \ldots, \lambda^{a_{1n}} t_n; \lambda^{b_{11}} x_1, \ldots, \lambda^{b_{1m}} x_m) \\ ((1, \mu), (t_1, \ldots, t_n; x_1, \ldots, x_m)) &\longmapsto (\mu^{a_{21}} t_1, \ldots, \mu^{a_{2n}} t_n; \mu^{b_{21}} x_1, \ldots, \mu^{b_{2m}} x_m) \end{aligned}$$

Example 2.1. In the case where $a_{1i} = 1$, $a_{2i} = 0$ for all $1 \leq i \leq n$ and $b_{1j} = 0$, $b_{2j} = 1$ for all $1 \leq j \leq m$, we have that $\mathbb{F}_M \cong \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$.

Example 2.2. In the case where $a_{2i} = 0$ for all $1 \leq i \leq n$ and $b_{1j} = 0$, $b_{2j} = 1$ for all $1 \leq j \leq m$, we have that $\mathbb{F}_M \cong \mathbb{P}(a_{11}, \cdots, a_{1n}) \times \mathbb{P}^{n-1}$.

Example 2.3. In the case where M is of the form

$$M = \left(\begin{array}{cc|cccc} 1 & 1 & -b_1 & -b_2 & \cdots & -b_m \\ 0 & 0 & 1 & 1 & \cdots & 1 \end{array}\right) \in \mathrm{Mat}_{2\times(2+m)}(\mathbb{Z})$$

for some $b_i \in \mathbb{Z}$, we will denote $\mathbb{F}_M$ by $\mathbb{F}(b_1, b_2, \ldots, b_m)$ [Rei962, Section 2.2].

Example 2.4. For $n > 0$ the surface $\Sigma_n = \mathbb{F}(n, 0)$ is known as the ***n*-th Hirzebruch surface**.

3 Elliptic fibrations

As the title of the project hints, we will be interested in surfaces with a very particular structure.

Definition 3.1. *A **genus** 1 **fibration** on a smooth projective variety X is a surjective proper morphism to a variety Z, $f : X \to Z$, such that all fibers except for finitely many are smooth curves of genus 1.*

Given that an elliptic curve is a genus 1 curve with a point, in order to construct an elliptic fibration, we need to find a method to set a point in each fibre in a way that we can still work consistently with the group law of the elliptic curve in each fibre. The way we can achieve this is by finding what is known as a section.

Definition 3.2. *Let $f : X \to Z$ be a morphism of algebraic varieties. A **section** of f is a morphism*

$$\sigma : Z \longrightarrow X$$

such that the composition

$$f \circ \sigma : Z \longrightarrow Z$$

is the identity map on Z.

Then,

Definition 3.3. *An **elliptic fibration** is a genus 1 fibration $f : X \to Z$ with a section.*

3.1 Elliptic surfaces

In this project we will be particularly interested in the cases of fibrations where X is a surface, for which we will need a few more hypothesis.

Definition 3.4. *An **elliptic surface** is a genus 1 fibration $f : X \to C$ from a smooth projective surface X to a smooth projective curve C with a section $\sigma_0 : C \to X$ which is **relatively minimal**, that is, that no fibre contains a (-1)-curve.*

The classical theory of elliptic surfaces deals with surfaces defined over an algebraically closed field. We will also want to look at other fields, such as $k = \mathbb{Q}$. We will say that an elliptic surface X over C is defined over k if both X and C are defined over k and both of the maps

$$f : X \longrightarrow C \qquad \text{and} \qquad \sigma : C \longrightarrow X$$

are defined over k.

Example 3.1. The surface defined by the equation

$$sy^2z = x(x-z)(sx-tz) \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^1_{s,t}$$

is an elliptic surface over $\mathbb{Q}$ with a fibration to $\mathbb{P}^1$ given by the projection to the second factor and section given by

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \\ [s:t] &\longrightarrow ([0:1:0],[s:t]). \end{aligned}$$

Let X be an elliptic surface over C defined over k. We would like to associate to X an elliptic curve $\mathcal{E}/K$ where $K = k(C)$. Conversely, to each elliptic curve $\mathcal{E}/K$ we would want to assign a birational equivalence class of elliptic surfaces.

The way we can do this is the following:

Any section $\sigma : C \to X$ of f defines a curve $D = \sigma(C)$ inside X which meets every fibre transversally in a single point. The curve D extends naturally to the underlying scheme $\mathcal{X}$ by taking the Zariski closure; thus it meets the generic fibre in a single K-rational point.

Conversely, given any $P \in \mathcal{E}(K)$, the closure of P in $\mathcal{X}$ is a curve in X. Let D denote the restriction to the elliptic surface X. By construction, D is endowed with a finite, birational morphism

$$f|_D : D \longrightarrow C.$$

Since C is smooth this is an isomorphism of D onto C. Thus the inverse of $f|_D$ gives a unique section $\sigma : C \longrightarrow X$ with $\mathrm{Im}(\sigma) = D$.

Example 3.2. Going back to our previous example, the point $[x_0 : y_0 : z_0] \in \mathcal{E}(\mathbb{Q}(t))$ would correspond to the section

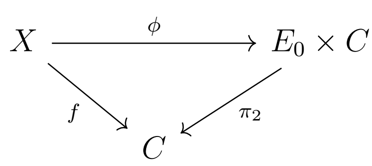
$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \\ [s:t] &\longrightarrow ([x_0:y_0:z_0],[s:t]). \end{aligned}$$

When we work with elliptic surfaces, we want to discard as examples of fibrations those who are "constant", in the sense that their fibres do not change with C. More rigorously,

Definition 3.5. *An elliptic surface $X \to C$ **splits** (over k) if there is an elliptic curve E_0/K and an isomorphism*

$$\phi : X \xrightarrow{\sim} E_0 \times C$$

such that the following diagram commutes



When we restrict ourselves to working with non-split elliptic surfaces, some of the theorems that have been proven for elliptic curves over number fields still work for function fields. In particular, we have,

Theorem 3.1 (Mordell-Weil theorem for function fields). *Let $X \to C$ be an elliptic surface defined over a field k, and let $\mathcal{E}/K$ be the corresponding elliptic curve over the function field $K = k(C)$. If $X \to C$ does not split, then $\mathcal{E}(K)$ is a finitely generated group* [Sil94, III, Theorem 6.1].

When working with fibrations defined over $\mathbb{P}^1$, the fact that a surface is non-split implies that the determinant of $\mathcal{E}(k(t))$ is a non constant rational function (which, with a change of coordinates becomes a polynomial). The places where this polynomial become zero therefore correspond to fibres that are not genus 1 curves, hence, singular curves. These are known as **singular fibres** and, as we mentioned, their existence is guaranteed in the case of non-split elliptic surfaces defined over $\mathbb{P}^1$ (though this is not the case for fibrations over curves with genus greater than 1, as Schütt and Shioda argue [SS097, Section 4.9]).

3.2 Singular fibres

From now on, we will focus in the case of non-split elliptic surfaces whose fibration is defined over $\mathbb{P}^1$. Let $F_v = f^{-1}(v)$ be a singular fibre. We can always write it as a divisor

$$F_v = \sum_{i=0}^{m_v-1} \mu_{v,i}\Theta_{v,i}$$

where

- m_v is the number of distinct irreducible components in F_v.
- $\Theta_{v,i}$ are the irreducible components.
- $\mu_{v,i}$ is the multiplicity of $\Theta_{v,i}$ in F_v.

We can arrange the ordering of the irreducible fibre components $\Theta_{v,i}$ as in the following theorem [SS19, Theorem 5.11].

Theorem 3.2. *The singular fibres satisfy the following:*

1. *There exists a unique component of F_v which intersects the zero section O, known as the* ***identity component*** *and denoted by $\Theta_{v,0}$, which has $\mu_{v,0} = 1$.*

2. *If F_v is an irreducible singular fibre, then $\Theta_{v,0}$ is either a rational curve with a node (type I_1) or a rational curve with a cusp (type II).*

3. *If F_v is a reducible singular fibre ($m_v > 1$), then every component $\Theta_{v,i}$ of F_v is a smooth rational curve which has self-intersection number $(\Theta_{v,i})^2 = -2$.*

We then have the following classification of the reducible singular fibres [SS19, Theorem 5.12].

Theorem 3.3. *For simplicity, let's write m instead of m_v and Θ_i instead of $\Theta_{v,i}$. Then, every possible reducible singular fibre can be classified into one of these types:*

$$I_m, III, IV, I_b^*, IV^*, III^*, II^*.$$

where $m > 1$ and $b \geq 0$. The fibre configurations are the following:

$$\boxed{I_m : \quad F_v = \Theta_0 + \cdots + \Theta_{m-1}}$$

where for $m = 2$ the two components intersect transversely at two points so that $\Theta_0 \cdot \Theta_1 = 2$ and where for $m \geq 3$, $\Theta_i \cdot \Theta_{i+1} = \Theta_{m-1} \cdot \Theta_0 = 1$ for all $0 \leq i < m$.

$$\boxed{III : \quad F_v = \Theta_0 + \Theta_1}$$

where $m = 2$ and the two components intersect at a single point with $\Theta_0 \cdot \Theta_1 = 2$.

$$\boxed{IV : \quad F_v = \Theta_0 + \Theta_1 + \Theta_2}$$

where $m = 3$ and the three components meet at a single point and the intersection numbers are $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_0 = 1$.

$$\boxed{I_b^* : \quad F_v = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4 + \cdots + 2\Theta_{b+4}}$$

where $m = b+5$, $\Theta_0 \cdot \Theta_4 = \Theta_1 \cdot \Theta_4 = \Theta_2 \cdot \Theta_{b+4} = \Theta_3 \cdot \Theta_{b+4} = 1$ and we also have that $\Theta_4 \cdot \Theta_5 = \Theta_5 \cdot \Theta_6 = \cdots = \Theta_{b+3} \cdot \Theta_{b+4} = 1$.

$$\boxed{IV^* : \quad F_v = \Theta_0 + \Theta_1 + 2\Theta_2 + 3\Theta_3 + 2\Theta_4 + \Theta_5 + 2\Theta_6}$$

where $\Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_3 = \Theta_3 \cdot \Theta_4 = \Theta_4 \cdot \Theta_5 = \Theta_3 \cdot \Theta_6 = \Theta_6 \cdot \Theta_0 = 1$ and $m = 7$.

$$\boxed{III^* : \quad F_v = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + \Theta_6 + 2\Theta_7}$$

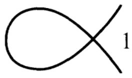
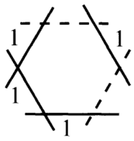
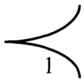
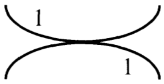
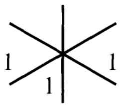
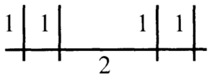
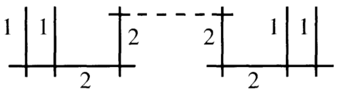
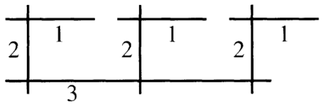
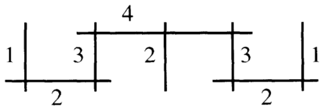
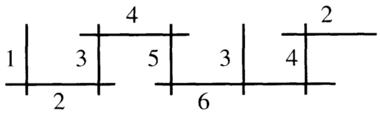
where $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_3 = \Theta_3 \cdot \Theta_4 = \Theta_4 \cdot \Theta_5 = \Theta_5 \cdot \Theta_6 = \Theta_3 \cdot \Theta_7 = 1$ and $m = 8$.

$$\boxed{II^* : \quad F_v = \Theta_0 + 2\Theta_7 + 3\Theta_6 + 4\Theta_5 + 5\Theta_4 + 6\Theta_3 + 4\Theta_2 + 2\Theta_1 + 3\Theta_8}$$

where $\Theta_0 \cdot \Theta_7 = \Theta_7 \cdot \Theta_6 = \Theta_6 \cdot \Theta_5 = \Theta_5 \cdot \Theta_4 = \Theta_4 \cdot \Theta_3 = \Theta_3 \cdot \Theta_2 = \Theta_2 \cdot \Theta_1 = \Theta_3 \cdot \Theta_8 = 1$ and $m = 9$.

For any other pair $i < j$ we have that $\Theta_i \cdot \Theta_j = 0$, i.e. Θ_i and Θ_j are disjoint.

This information can be summarised in the next table:

Type	m_v	Configuration
I_0	1	
I_1	1	
I_n	n	
II	1	
III	2	
IV	3	
I_0^*	5	
I_n^*	$n+5$	
IV^*	7	
III^*	8	
II^*	9	

Once again, we have a connection between these singular fibres and the ADE-classification that we have explored in past sections, which is given by the restricted dual graph.

The **dual graph** of a reducible fibre F_v with m irreducible components Θ_i consists of m vertices and several edges connecting them with a number indicating the intersection number of the components if they intersect each other. Then, by the **restricted dual graph** of a reducible fibre F_v, we mean the subgraph of the dual graph obtained by deleting the vertex Θ_0 (and the edges starting from it).

We therefore have,

Proposition 3.1. *The restricted dual graph of a reducible fibre is the same as the Dynkin diagram for the root lattices of types A_n, D_n, E_n. Furthermore, if we denote by T_v the lattice of rank $m-1$ generated by the Θ_i with the intersection pairing, T_v is a negative-definite lattice isomorphic to a root lattice of type A_n^-, D_n^- or E_n^- according to the following table*

Type of F_v	$I_m(m>1)$	III	IV	$I_b^*(b \geq 0)$	IV^*	III^*	II^*
Root Lattice	A_{m-1}^-	A_1^-	A_2^-	D_{b+4}^-	E_6^-	E_7^-	E_8^-

The next natural question involves how to compute the type of the singular fibres of an elliptic fibration. The idea is that given an elliptic curve defined over the function field of a curve C, we can always find a change of coordinates to a generalised Weierstrass equation in $\mathbb{P}^2$ of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Given a singular fibre, by performing blow-ups and blow-downs according to a procedure known as **Tate's algorithm**, we can find a model that is locally minimal at that singular fibre. Once we have that minimal model, we can easily read the type of singular fibre from the a_i coefficients. The details of this algorithm can be found in the works of Schütt and Shioda [SS097, Section 4.2], and Silverman [Sil94, IV, Section 9].

3.3 Lattices of elliptic surfaces

Previously, we claimed that studying the lattice $\mathrm{Num}(X)$ of a surface was a useful method for understanding it geometrically. In the case of elliptic surfaces, this lattice will also give us information about its associate elliptic curve over a function field.

The first comment that must be done about elliptic surfaces is that their Néron-Severi group is always torsion-free [SS19, Theorem 6.4]. That is, algebraic and numerical equivalence coincide, so we can talk about their **Néron-Severi lattice**.

Not only that, but in terms of equivalence, when we work with a fibration there is some sort of uniformity in the fibres. If $f: X \to C$ is an elliptic fibration, two fibres are linearly equivalent if and only if either they are the same or the base curve C has genus zero. Regardless of the base curve C, any two fibres are algebraically equivalent.

Additionally, the fibration sets certain restrictions on the geometry of X. For instance, $K_X^2 = 0$ and if we let F denote any smooth fibre of an elliptic fibration then, $F^2 = 0$. For that reason, when we are working with the Néron-Severi group of an elliptic surface X it is important to separate the contributions of the divisors of the fibration from the other divisors.

Definition 3.6. *The* ***trivial lattice*** $\operatorname{Triv}(X)$ *if the sublattice of* $\operatorname{NS}(X)$ *generated by the zero section* O *and fibre components. More explicitly,*

$$\operatorname{Triv}(X) = \langle O, F\rangle \oplus \bigoplus_{v\in R} T_v,$$

where R *denotes the finite subset* $\{v \in C(k) | T_v \neq 0\}$ *of points on the base curve where the singular fibres are located.*

It is then easy to check that

$$\langle O, F\rangle = \begin{pmatrix} -\chi(X) & 1 \\ 1 & 0 \end{pmatrix}$$

and therefore

$$\rho(X) \geq \operatorname{rank}(\operatorname{Triv}(X)) = 2 + \sum_{v\in R}(m_v - 1)$$

The divisors that are part of the trivial lattices are known as **vertical divisors**. On the other hand, we have the divisors known as **horizontal divisors**, which are linear combinations of irreducible curves with integer coefficients where each curve meets any fibre with a given and fixed multiplicity $m > 0$. We also refer to such a curve as a **multisection** of the degree m, and if $m = 1$, we simply call them **sections**. This name is not casual, as it happens to be that these horizontal sections correspond to the sections of our elliptic fibration. More precisely, we can always express any divisor as a sum of a vertical and a horizontal divisor:

Theorem 3.4. *Let* $f : X \to C$ *be an elliptic surface with generic fibre* $\mathcal{E}(K)$ *over* $K = k(C)$*. The map that sends a point* $P \in \mathcal{E}(K)$ *to the horizontal component of its associate divisor is well-defined and defines an isomorphism of abelian groups*

$$\mathcal{E}(K) \cong \operatorname{NS}(X)/\operatorname{Triv}(S).$$

As a consequence, we have,

Corollary 3.1. *Let* X *be an elliptic surface and let* $r = \operatorname{rank}(\mathcal{E}(K))$*. Then,*

$$\rho(X) = r + 2 + \sum_{r\in R}(m_v - 1).$$

Theorem 3.4 has an important consequence, which is that if we work with the rational lattice $\operatorname{NS}(X)_{\mathbb{Q}} = \operatorname{NS}(X) \times_{\mathbb{Q}} \mathbb{Q}$, we can perfectly define the horizontal divisors in a unique way inside of this space, obtaining a group homomorphism

$$\phi : \mathcal{E}(K) \longrightarrow \operatorname{NS}(X)_{\mathbb{Q}}$$

whose kernel is $\ker(\varphi) = \mathcal{E}(K)_{\text{tors}}$. From the intersection pairing in $\operatorname{NS}(X)_{\mathbb{Q}}$, we thus obtain a pairing in $\mathcal{E}(K)/\mathcal{E}(K)_{\text{tors}}$ known as the **height pairing**, which makes this group a lattice known as the **Mordell–Weil lattice** of $\mathcal{E}(K)$.

4 The theory of K3 surfaces

Some of the geometric properties that an elliptic surface X must satisfy, such as $K_X^2 = 0$ add strong restrictions to which kind of surfaces admits an elliptic fibration.

In terms of complexity, on the simpler side of the spectrum, we have **elliptic rational surfaces** which have been completely studied and classified, mostly due to the fact that they always have Picard rank 8 and their Néron-Severi lattice is E_8 [SS097, Chapters 7 and 8]. On the other side, we have elliptic surfaces with what it is known as Kodaira dimension $\kappa = 1$, called by some authors "**honestly elliptic surfaces**", for which we do not know much about their structures.

For this project, we decided to focus in an intermediate case and study **K3 surfaces**. Many results have been proven for these surfaces, yet there are still interesting questions to ask about them.

Definition 4.1. *A* ***K3 surface*** *is a compact surface X with*

$$h^1(X, \mathcal{O}_X) = 0 \qquad \textit{and} \qquad K_X = 0.$$

From the definition, we deduce that all K3 surfaces have $q(X) = 0$ and $p_g(X) = 1$. By Serre duality, and the definition of the Euler characteristic, we get that $\chi(X) = 2$. From Noether's formula, we then deduce that $e(X) = 12\chi(X) = 24$ and this give us the Betti numbers $b_0(X) = 1, b_1(X) = 0, b_2(X) = 22, b_3(X) = 0$ and $b_4(X) = 1$.
The Hodge diamond of a K3 can therefore easily computed to be:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Some examples of K3 surfaces are

- Smooth quartics in $\mathbb{P}^3$.
- Smooth intersections of degree $(2,3)$ in $\mathbb{P}^4$ or $(2,2,2)$ in $\mathbb{P}^5$.
- Double coverings $X \to \mathbb{P}^2$ branched along a smooth sextic curve. These can be understood as varieties in the weighted projective space $\mathbb{P}(3,1,1,1)$ of degree 6.

In general, using the methods explained in section 1.1 to compute the canonical divisor of surfaces in the weighted projective space $\mathbb{P}(a_1, a_2, a_3, a_4)$, we can deduce that for appropriate well-formed weights (a_1, a_2, a_3, a_4), a smooth hypersurface with degree $a_1 + a_2 + a_3 + a_4$ is a K3 surface. In 1979, Reid computed all possible 95 families of K3 surfaces that arise this way through an algorithm that has been described and improven by Brown and Kasprzyk [BK164, Section 2].

The following example of K3 surface is also relevant.

Definition 4.2. *Let A be the Jacobian of a smooth hyperelliptic curve of genus 2 and let $\iota : A \to A$ be the involution defined by $\iota(x) = -x$. The* ***Kummer surface*** *associated to A is the resolution of the surface defined as the quotient of A by ι, which has 16 singular points.*

For instance, if we take A to be isomorphic to the product of two elliptic curves

$$\mathcal{E}_1 : \quad y^2 = f(x), \qquad\qquad \mathcal{E}_2 : \quad s^2 = g(t),$$

an affine model of the Kummer surface of $\mathcal{E}_1 \times \mathcal{E}_2$ is given by the double covering

$$w^2 = f(x)g(t).$$

4.1 Elliptic K3 surfaces

We can now focus on the type of K3 surfaces that we are interested in, those who admit an elliptic fibration. Since any K3 surface X is simply connected by definition, the base curve of any elliptic fibration has to be $\mathbb{P}^1$, as any regular 1-form on the base pulls back to a regular 1-form on X. It is then possible to prove that X admits a globally minimal Weierstrass form

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6$$

where all the $a_i(t)$ are polynomials that satisfy that $\deg(a_i(t)) \leq 2i$ and that there is some i such that $\deg(a_i(t)) > 1$.

It is important to note as well that not all K3 surfaces are elliptic. In characteristic different than 2 and 3, there is an easy criterion to check if a K3 surface admits a genus 1 fibration:

Theorem 4.1. *A K3 surface X admits a genus 1 fibration if and only if there is a non-zero divisor $D \in \mathrm{NS}(X)$ with $D^2 = 0$* [PŠŠ716, Section 3, Theorem 1].

The thought behind the proof of this theorem is that, by adding and subtracting (-2)-curves to D, we can always find a divisor D' such that $(D')^2 = 0$ and $|D'|$ is base point free. From the adjunction formula, we deduce that

$$2g(D') - 2 = K_X \cdot D' + (D')^2 = 0 + 0 = 0,$$

and therefore D' has genus 1, and $|D'|$ induces the elliptic fibration.

To upgrade this condition to get a K3 surface to admit an elliptic fibration, we need the next result:

Theorem 4.2. *Let X be a K3 surface with a genus one fibration induced by a non-zero divisor D with $D^2 = 0$, as in theorem 4.1. Then any divisor $C \in \mathrm{Div}(X)$ with $D \cdot C = 1$ gives a section of the fibration, therefore making such fibration elliptic* [SS19, Proposition 11.31].

Although the last two theorems are good to construct explicit fibrations, in order to check if a K3 surface admits a fibration, we have a nicer condition:

Theorem 4.3. *Any K3 surface X with Picard number $\rho(X) \geq 5$ admits an elliptic fibration* [LSY13, Section I.3.4].

4.2 The K3 lattice

From the fact that the second Betti number of a K3 surface X is 22 and $h^{1,1}(X) = 20$, we deduce that in the characteristic zero case, the Picard rank must be less or equal than 20 and in positive characteristic, it can be 22 at most. Also, the Hodge index theorem implies that the Néron-Severi lattice of a K3 surface must have signature $(1, \rho(X) - 1)$.

These two ideas already give us some information about $\mathrm{NS}(X)$, but the biggest contribution to understanding this lattice comes from the fact that $H^2(X, \mathbb{Z})$ can be seen as a lattice endowed with the cup product and that, regardless of the X considered, this lattice is always the same up to isometry. The lattice $H^2(X, \mathbb{Z})$ is known as the **K3 lattice**, has signature $(3, 19)$ and presents the following structure [LSY13]:

Proposition 4.1. *The **K3 lattice** is the even unimodular lattice*

$$L_{K3} = U \oplus U \oplus U \oplus E_8^{-1} \oplus E_8^{-1}$$

where U is the 2-dimensional lattice with Gram matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In positive characteristic the classification of all possible lattice configurations that can arise from the Néron-Severi group of a K3 surface is tricky due to the existence of inseparable base changes. However in characteristic zero, over an algebraically closed fields, these lattice configurations can be studied from the embedding $\mathrm{NS}(X) \hookrightarrow L_{K3}$.

Shimada carried the computations to find all possible configurations for elliptic surfaces in the complex case [Shi001]:

Theorem 4.4. *Among all possible complex elliptic K3 surfaces, there are 3693 distinct admissible configurations of pairs (Σ, G) where Σ is the sum of root lattices of ADE-type and G is a finite abelian group representing the torsion of the Mordell-Weil group of one of these K3 surfaces.*

Despite knowing the classification of all elliptic K3 surfaces, it is still a challenge to find examples of each configuration. Given a K3 surface, it is not an easy task to determine its geometric information either, as we will attempt in the next section.

5 Study of an elliptic K3 surface

The goal of this section is to apply some of the techniques expressed in the previous sections to understand the geometry and of some examples.

Let X be the hypersurface of $\mathbb{P}^3_{x,y,z,w}$ given by the equation

$$\boxed{f: \quad x^3y + xy^2w + zyw^2 + z^3w = 0}$$

This surface appeared while attempting to find K3 hypersurfaces on the scroll

$$S_{u,v,x,y,z} : \left(\begin{array}{cc|ccc} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

By performing a search among all the possible hypersurfaces that could be expressed as the sum of four monomials with weight $\binom{-1}{3}$, one of the surfaces Y with the most reducible fibres was the one defined by the homogeneous weighted polynomial

$$g : x^3 + uvxy^2 + uv^2y^2z + u^2vz^3 = 0$$

which has du Val singularities E_8 in $[0:1\ ,\ 0:0:1]$, A_2 in $[1:0\ ,\ 0:1:0]$ and A_5 in $[1:0\ ,\ 1:0:0]$.

There is a birational map

$$\begin{aligned} Y &\longrightarrow X \\ [u:v\ ,\ x:y:z] &\longmapsto [x:uy:uz:vy] \end{aligned}$$

that allows us to relate Y to the hypersurface X given by a degree 4 polynomial in $\mathbb{P}^3$. Since a birational map between K3 surfaces is automatically an isomorphism due to the fact that the canonical divisor is nef, we can study this surface in projective space, where we have more control over its geometry.

The gradient of X is

$$\nabla_{\{x,y,z,w\}} f = \{3x^2y + wy^2,\ x^3 + 2wxy + w^2z,\ w^2y + 3wz^2,\ xy^2 + 2wyz + z^3\}$$

By finding when all these factors are simultaneously zero, it can easily be checked that X is a singular surface with singular points $[0:1:0:0]$ and $[0:0:0:1]$. Either by performing successive blow-ups in these points or by finding a suitable transformation into a normal form, we can check that both of these singularities are du Val singularities of type A_8. In `Magma` [BCP979], both the singularities and the geometry of X can be studied through the code 7.1.

We would be like to answer some questions about X such as whether it admits an elliptic fibration or what its Picard rank is.

5.1 Elliptic fibrations

Let's recall that theorem 4.1 stated that any divisor with self-intersection number 0 induced a genus 1 fibration of X.

Our surface X has the special virtue that it contains many lines, so let us fix one of them: ℓ. Let H represent a hyperplane section in X. Then, $(H-\ell)^2 = 0$. Indeed, $H^2 = 4$, as the intersection of two hyperplanes in general position is a line in general position, which we know it intersects a degree 4 surface in exactly 4 points. We also know that $\ell^2 = -2$ either from the fact that we mentioned in section 1 that $\ell^2 = 2 - \deg(X) = -2$ or from the adjunction formula as $g(\ell) = 0$ and $H \cdot \ell = 1$. This can easily be seen by considering an H that contains ℓ, as then $H = \ell + C$ where C is a (possibly degenerated) cubic and we therefore have that

$$H \cdot \ell = \ell^2 + \ell \cdot C = -2 + 3 = 1.$$

Finally,

$$(H - \ell)^2 = H^2 - 2\, H \cdot \ell + \ell^2 = 4 - 2 - 2 = 0.$$

From the proof of theorem 4.1 we also know that, explicitly, this genus 1 fibration is given by the linear system $|H - \ell|$, which correspond to the pencil of hyperplanes in $\mathbb{P}^3$ that contain ℓ. Now, to find a section of this fibration, by theorem 4.2 we need to find a divisor E such that $(H - \ell) \cdot E = 1$. Our most two natural guesses, H and ℓ, do not work, as

$$(H - \ell) \cdot H = 3 \qquad\qquad (H - \ell) \cdot \ell = 3.$$

However, for this particular example, what we can do is to consider another line ℓ' that is disjoint to ℓ. Because $\ell \cap \ell' = \emptyset$, we have that

$$(H - \ell) \cdot \ell' = H \cdot \ell' - \ell \cdot \ell' = 1 - 0 = 1.$$

Now that we have seen that the theory works, we can work out the explicit construction. Let ℓ be the line defined by the equation $x = z = 0$. Then, the pencil of hyperplanes going through ℓ is given by $tx - sz = 0$, where $t, s \in k$ and both of them are not simultaneously zero. This give us a genus 1 fibration over $\mathbb{P}^1$ defined by

$$\begin{aligned} X &\xrightarrow{f} \mathbb{P}^1 \\ [x : y : z : w] &\longmapsto [s : t] \end{aligned}$$

where, outside of ℓ, we have that $[s : t] = [x : y]$. There is a section given by the line ℓ' defined by $y = w = 0$, as we have that

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\sigma} \ell' \subset X \\ [x : z] &\longmapsto [x : 0 : z : 0] \end{aligned}$$

and therefore $f \circ \sigma$ is the identity.

Computing the intersection of X with $tx - sz = 0$, we get that the fibres $X_{[s:t]}$ are given by

$$s^3x^3y + s^3xy^2w + s^2txyw^2 + t^3x^3w = tx - sz = 0$$

which, as $x \neq 0$, these are isomorphic to the family of genus 1 curves $\mathcal{X}_{[s:t]}$ in $\mathbb{P}^2_{x,y,w}$ with coefficients in $k[s,t]$:

$$s^3x^2y + s^3y^2w + s^2tyw^2 + t^3x^2w = 0$$

In every fibre, the section that we constructed corresponds to the point of $X_{[s:t]}$:

$$[x : y : w] = [1 : 0 : 0]$$

After dehomogenising the equations with respect to s and setting $[1 : 0 : 0]$ to be the point of infinity, if $\mathrm{char}(k) \neq 2, 3$ this elliptic fibration can then be rewritten in minimal Weierstrass form as the elliptic curve $\mathcal{E}/k(t)$:

$$Y^2 = X^3 + (-\tfrac{1}{3}t^6 + \tfrac{1}{3}t^4 - \tfrac{1}{3}t^2)X + (-\tfrac{2}{27}t^9 + \tfrac{1}{9}t^7 + \tfrac{1}{9}t^5 - \tfrac{2}{27}t^3)$$

where the change of coordinates is given by

$$[x : y : w] \longmapsto (X, Y) = \Big(\frac{(\frac{2}{3}t^3 - \frac{1}{3}t)y + (-\frac{1}{3}t^6 + \frac{2}{3}t^4)w}{y + t^3w}, \frac{(t^6 - t^4)x}{y + t^3w}\Big). \qquad (1)$$

If $\mathrm{char}(k) = 3$, a similar change of coordinates can be found to obtain the minimal model form:

$$Y^2 = X^3 + (2t^3 + 2t)X^2 + t^4X.$$

As a sanity check, we can observe that regardless of the characteristic of the field, the equation of $\mathcal{E}/k(t)$ corresponds to a possible minimal equation of an elliptic K3 surface, as if we express it in the form:

$$Y^2 = X^3 + a_2(t)X^2 + a_4(t)X + a_6(t)$$

all the coefficients satisfy that $\deg(a_i(t)) \leq 2i$ and $\deg(a_i(t)) > i$ for some value of i.

We can now study some aspects of this fibrations. Regardless of the characteristic, these elliptic curves always have full 2-torsion, as we always have in $\mathrm{char}(k) \neq 2, 3$ the generators of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$P_1 = (\tfrac{1}{3}\, t\,(2t^2 - 1), 0), \qquad P_2 = (\tfrac{1}{3}\, t\,(-t^2 + 2), 0), \qquad P_3 = (\tfrac{1}{3}\, t\,(-t^2 - 1), 0).$$

and in $\mathrm{char}(k) = 3$:

$$P_1 = (t, 0), \qquad P_2 = (t^3, 0), \qquad P_3 = (0, 0).$$

As we can compute the inverse of (1),

$$(X,Y) \longmapsto [x:y:w] = [3Y : t^6 - 2t^4 + 3t^3X : 2t^3 - t - 3X], \tag{2}$$

we can easily check that P_1 correspond to the image of the singular point $[0:1:0:0]$, P_2 corresponds to the image of the singular point $[0:0:0:1]$ and P_3 corresponds to the image of $[0:t:0:-s]$, which is the third point of intersection of $\mathcal{X}_{[s:t]}$ with ℓ.

Using `Magma` (code 7.2), we can analyse other aspects of this fibration. For instance, the discriminant and the *j*-invariant are

$$\Delta(\mathcal{E}) = 16t^{10}(t-1)^2(t+1)^2, \qquad j(\mathcal{E}) = \frac{256(t^4-t^2+1)^3}{t^4(t-1)^2(t+1)^2}.$$

In characteristic different than zero, this fibration always has 4 singular fibres:

- A split multiplicative singular fibre at $t=1$ of type I_2 with multiplicity 2 in the discriminant and multiplicity 1 in the conductor.
- A multiplicative singular fibre at $t=-1$ of type I_2 with multiplicity 2 in the discriminant and multiplicity 1 in the conductor. Applying Tate's algorithm, we can see that this fibre splits if and only if -1 is a square in k.
- An additive singular fibre at $t=0$ of type I_4^* with multiplicity 10 in the discriminant and multiplicity 2 in the conductor.
- An additive singular fibre at $t=\infty$ of type I_4^* with multiplicity 10 in the discriminant and multiplicity 2 in the conductor.

In the database of K3 elliptic surfaces in the Graded Ring Database, this fibration correspond to the element 2471 [Shi001].

The number of reducible components of the fibre type I_4^* is 9 and the number of reducible components of I_2 is 2. Therefore, if $r = \operatorname{rank}(\mathcal{E}(k(t))$, we have that

$$\rho(X) = 2 + r + 2(9-1) + 2(2-1) = 20 + r$$

As for fields with $\operatorname{char}(k) = 0$ the Picard rank of any K3 surface is bounded by 20, we deduce that $\operatorname{rank}(\mathcal{E}(k(t)) = 0$. In particular, we can deduce that

$$\mathcal{E}(\mathbb{Q}(t)) = \{O, (\tfrac{1}{3}\,t\,(2t^2-1), 0),\ (\tfrac{1}{3}\,t\,(t^2-2), 0),\ (\tfrac{1}{3}\,t\,(-t^2-1), 0)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

5.2 Study of the Picard lattice of X

From what we deduced from the previous section, the trivial lattice of X with respect to the fibration is

$$\operatorname{Triv}(X) = U \oplus A_1^{-1} \oplus A_1^{-1} \oplus D_8^{-1} \oplus D_8^{-1}$$

which has matrix:

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

The hyperbolic plane U component has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

which corresponds to the intersection matrix of $H - \ell$ and ℓ', where

$$\ell : \quad x = z = 0, \qquad\qquad \ell' : \quad y = w = 0.$$

Using the inverse map (2), we can compute that the divisor which degenerates to the singular fibre at $t = 1$ is the line

$$\ell_2 : \quad x - z = y + w = 0,$$

and that the divisor that degenerates to the singular fibre at $t = -1$ is the line

$$\ell_3 : \quad x + z = y - w = 0.$$

On the other hand, the divisors that degenerate to the fibres at $t = 0$ and $t = \infty$ are, respectively,

$$\ell_4 : \quad y = z = 0, \qquad\qquad \ell_4 : \quad x = w = 0.$$

Each of these divisors contains one of the singularities of X, which were of type A_8 and whose resolution comprises 8 exceptional curves. This motivates where the I_4^* singularity come from, as all the 8 exceptional curves get compressed into a singular fibre.

The theory of elliptic fibrations has helped us find the Picard rank quite easily, but suppose we were studying a K3 surface without elliptic fibrations (which it is possible for K3 surfaces with Picard rank smaller than 5).

We can still compute a lower bound for the Picard rank using the following strategy: we can enumerate divisors of our surface and compute their intersection matrix to find a set of independent elements of the Néron-Severi group. The size of this set is the lower bound we are looking for.

For the X we are working with, we can easily compute many independent divisors, as it is reflected in the `Magma` code (7.3). In fact, it is easy to check that the following 4 divisors are independent

$$\begin{aligned} \ell :&\quad y = w = 0 & \qquad \ell_2 :&\quad x - z = y + w = 0 \\ \ell_3 :&\quad x + z = y - w = 0 & \qquad D_6 :&\quad x - z = x^2 + yw = 0 \end{aligned}$$

As none of these divisors contains a singular point, we deduce the existence of 20 independent divisors in X, as we have these 4 plus the two sets of 8 exceptional lines that we get when we blow-up the A_8 singularities. By the same argument that $\rho(X) \leq 20$, we are able to prove that in characteristic zero, those divisors generate the Néron-Severi lattice of X, which has matrix:

$$\begin{pmatrix}
-2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2
\end{pmatrix}$$

5.3 Finding an upper bound of the Picard number

In characteristic zero we have many interesting tools to analyse K3 surfaces, mainly given by Hodge theory and complex geometry. However, interestingly enough, working in positive characteristic is precisely what will enable us to compute an upper bound of $\rho(X)$ by counting points on X over finite fields. This strategy has been used by many to solve problems such as finding K3 surfaces with Picard number one and infinitely many points [vL072] or elliptic K3 surfaces with Mordell-Weil rank 15 [Klo076].

Let $k \cong \mathbb{F}_q$ for $q = p^r$ and let $\overline{X} = X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ and $\widetilde{X_p} = X \times_k \overline{k}$. We assume that $\overline{X}$ and $\widetilde{X_p}$ are integral projective surfaces. Let $l \neq p$ be a prime number. For any scheme Z over k, we can define

$$H^2_{\text{ét}}(Z, \mathbb{Q}_l) = \left(\varprojlim H^2_{\text{ét}}(Z, \mathbb{Z}/l^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Furthermore, for every integer m and every vector space H over $\mathbb{Q}_l$ with the Galois group $G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acting on it, we define the **twistings** of H to be the $G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-spaces $H(m) = H \otimes_{\mathbb{Q}_l} W^{\oplus m}$, where

$$W = \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \left(\varprojlim \mu_{l^n} \right)$$

is the one-dimensional l-adic vector space on which $G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ operates according to its action on the group $\mu_{l^n} \subset \overline{\mathbb{F}_q}$ of l^n-th roots of unity. By setting $W^{\otimes 0} = \mathbb{Q}_l$ and, for $m < 0$, $W^{\otimes m} = \text{Hom}(W^{\otimes -m}, \mathbb{Q}_l)$; for a surface Z over $\overline{\mathbb{F}_q}$, the cup product gives $H^2_{\text{ét}}(Z, \mathbb{Q}_l)(m)$ the structure of an inner product space for all integers m.

We then have the following result [vL073, Proposition 6.2].

Proposition 5.1. *There exist natural injective homomorphisms*

$$\text{NS}(\overline{X}) \otimes_{\mathbb{Q}_l} \mathbb{Q}_l \hookrightarrow \text{NS}(\widetilde{X_p}) \otimes_{\mathbb{Q}_l} \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)$$

of finite dimensional vector spaces over $\mathbb{Q}_l$ such that the second injection respects the Galois action of $G(\overline{k}/k)$.

For any variety Z, let F_Z denote the absolute Frobenius which acts as the identity on points and as $f \mapsto f^p$ on the structure sheaf. Let $\varphi = F^r_{X_k}$ and let φ^*_2 denote the automorphism on $H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)$ induced by $\varphi \times 1$ acting on $X_k \times_k \overline{k} \cong \widetilde{X_p}$. It is then possible to prove that this φ^*_2 induces a homomorphism $\varphi^{*(1)}$ in $H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)$.

From proposition 5.1, it can be shown [vL073, Corollary 6.3] that the image of $\text{NS}(\widetilde{X_p})$ inside of $H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)$ lies in $H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)^{\varphi^{*(1)}}$, from which we deduce that

$$\rho(\widetilde{X_p}) \leq \dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)^{\varphi^{*(1)}}$$

In fact, there is a conjecture which states that, generally, these two numbers match:

Conjecture 5.1 (Tate's conjecture). *For any smooth projective surface over $\mathbb{F}_q$, one has*

$$\rho(\widetilde{X_p}) = \dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)^{\varphi^{*(1)}}.$$

Fortunately for us, Tate's conjecture has been proven for ordinary K3 surfaces over fields in all possible characteristics [Cha13, MP131, Mau1410, KMP1512]. This theory gives us a way of computing $\rho(\widetilde{X_p})$, as $\dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)(1)^{\varphi^{*(1)}}$ is equal to the number of eigenvalues λ of φ_2^* for which λ/q is a root of unity, counted with multiplicity.

In terms of computability, the first challenge is to calculate the characteristic polynomial associated to φ_2^*. In order to do so, we need to make use of Lefschetz's trace formula and the Weil conjectures [FK88, Chapter IV]. Let φ_i^* denote the induced automorphism of φ on $H^i_{\text{ét}}(\widetilde{X_p}, \mathbb{Q}_l)$. Then, the characteristic polynomial of $(\varphi_i^*)^n$ acting on $H^i(\widetilde{X_p}, \mathbb{Q}_l)$ is

$$f_i(t) = \det\left(\mathrm{Id} - t(\varphi_i^*)^n | H^i(\widetilde{X_p}, \mathbb{Q}_l)\right) = \prod_{i=1}^{b_i}(1 - \alpha_{ij}t).$$

From the Weil conjectures, we know that $f_i(t)$ is a rational polynomial and the reciprocal roots have absolute value $|\alpha_{ij}| = p^{ni/2}$.

Additionally, from X being a K3 surface, we know that the Betti numbers are:

i	0	1	2	3	4
b_i	1	0	22	0	1

Thus, from the Weil conjectures, we deduce the following information:

i	0	1	3	4
$f_i(t)$	$1-t$	1	1	$1-p^2t$
$\mathrm{Tr}\,\varphi_i^*$	1	0	0	p^2
$\mathrm{Tr}(\varphi_i^*)^n$	1	0	0	p^{2n}

From the last row we can compute the traces of powers of φ_2^* by applying the Lefschetz formula

$$\#X_p(\mathbb{F}_{p^n}) = \sum_{i=0}^{4}(-1)\,\mathrm{Tr}(\varphi_i^*)^n,$$

which gives us the expression

$$\mathrm{Tr}(\varphi_2^*)^n = \#X_p(\mathbb{F}_{p^n}) - p^{2n} - 1.$$

Finally, in order to find the characteristic polynomial of φ_2^*, we can use this result [vL072, Lemma 2.4.]:

Lemma 5.1. Let V be a vector space of dimension n and L a linear operator on V. Let l_i denote the trace of L_i. Then, the characteristic polynomial of L is equal to

$$f_L(t) = \det(t \cdot \mathrm{Id} - T) = t^n + c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$$

where the c_i are given recursively by

$$c_1 = -l_1 \qquad \text{and} \qquad -kc_k = l_k + \sum_{i=1}^{k-1} c_i l_{k-i}.$$

To sum up all these results, we have deduced that for any prime p, if we are able to compute the number of points of X_p over the field $\mathbb{F}_{p^n}$ for $1 \leq n \leq 22$, we will be able to compute the characteristic polynomial of φ_2^* and, subsequently, $\rho(\widetilde{X_p})$ as the number of eigenvalues λ of for which λ/p is a root of unity.

In principle, computing $\#X_p(\mathbb{F}_{p^n})$ for large n almost seems an impossible task. For reference, $2^{22} \approx 4.2 \times 10^6$, so if we tried a brute force approach of checking if each point $[x : y : z : w] \in \mathbb{P}^3$ over $\mathbb{F}_{2^{22}}$ belonged to X_2, we would have to check more than 7.3×10^{19} points!

Fortunately, the Weil conjectures tell us that the characteristic polynomial $f_2(t)$ of X_p satisfies the functional equation

$$p^{22} f_2(t) = \pm t^{22} f_2(p^2/t)$$

which implies that we only need to compute $\#X_p(\mathbb{F}_{p^n})$ up to $n = 11$, which is a great improvement.

Let's showcase this method for $p = 2$ on our favourite K3 surface

$$X: \quad x^3y + xy^2w + zyw^2 + z^3w = 0 \ \subset \mathbb{P}^4.$$

Using `Magma` we can compute all points of X in $\mathbb{F}_{2^n}$ up to $n = 10$ to get

n	1	2	3	4	5	6	7	8	9	10
$\#X(\mathbb{F}_{2^n})$	11	29	89	305	1121	4289	16769	66305	263681	1051649
$\mathrm{Tr}(\varphi_2^*)^n$	6	12	24	48	96	192	384	768	1536	3072
c_n	-6	12	-8	0	0	0	0	0	0	0

Due to technical difficulties, I have not been able to compute $\#X(\mathbb{F}_{2^{11}})$. However, in this case (and in many situations, in fact), knowing this quantity is not really necessary as we can infer its value from the information we know about the characteristic polynomial.

First, let's suppose that the sign in the functional equation is minus, so that

$$2^{22} f_2(t) = -t^{22} f_2(4/t).$$

This necessarily implies that $c_{11} = 0$. From all the other c_i that we know, we deduce that

$$f_2(t) = t^{22} - 6t^{21} + 12t^{20} - 8t^{19} + 524288t^3 - 3145728t^2 + 6291456t - 4194304$$

Now, if λ is a root of $f_2(t)$, then $\lambda/2$ is a root of $g(t) = 2^{-22} f_2(2t)$. As

$$\begin{aligned} g(t) &= 2^{-22} f_2(2t) \\ &= t^{22} - 3t^{21} + 3t^{20} - t^{19} + t^3 - 3t^2 + 3t - 1 \\ &= (t-1)^3(t^{19}+1) \end{aligned}$$

we deduce that all the roots of $g(t)$ are roots of unity. Therefore, assuming that the sign of the functional equation is minus, $\rho(\widetilde{X_2}) = 22$.

Now, let's suppose that the sign of the functional equation was a plus. From the functional equation we can only deduce that

$$f_2(t) = t^{22} - 6t^{21} + 12t^{20} - 8t^{19} + c_{11}t^{11} - 524288t^3 + 3145728t^2 - 6291456t + 4194304$$

for some $c_{11} \in \mathbb{Z}$. We then have

$$\begin{aligned} g(t) &= 2^{-22} f_2(2t) \\ &= t^{22} - 3t^{21} + 3t^{20} - t^{19} + \tfrac{1}{2048} c_{11} t^{11} - t^3 + 3t^2 - 3t + 1 \end{aligned}$$

However, as we know that 1 must always be a root of $g(t)$, we therefore deduce that $g(1) = \frac{1}{2048} c_{11} = 0$ and so,

$$\begin{aligned} g(t) &= 2^{-22} f_2(2t) \\ &= t^{22} - 3t^{21} + 3t^{20} - t^{19} - t^3 + 3t^2 - 3t + 1 \\ &= (t-1)^3(t^{19}-1) \end{aligned}$$

Hence, regardless of the sign of the functional equation, we deduce that

$$\rho(\widetilde{X_2}) = 22.$$

We also deduce that

$$\rho(\overline{X}) \leq \rho(\widetilde{X_2}) = 22$$

which, in this example, does not say much, but it is often a useful bound in the cases where we are studying a K3 surface with low Picard rank. In those cases, combining the bounds $\rho(\widetilde{X_p})$ that we get for different primes p, we can usually get an upper bound that is quite close to the actual Picard rank of $\overline{X}$.

What it is quite interesting for this example X is that both in characteristic zero and in characteristic 2, our surface has the greatest Picard rank possible (20 and 22 respectively). In characteristic zero the surfaces with maximal Picard rank are called **extremal** or singular, whereas in positive characteristic they are known as **supersingular surfaces**.

This supersingularity condition has been proven to be equivalent to the fact that the height of the formal Brauer group $\widehat{\mathrm{Br}}(X)$ is infinite [SS19, Section 12.4.1]. In the case where X is defined over a finite field, then the height is visible in the Newton polygon of the characteristic polynomial of φ_2^* [SS19, Remark 11.21].

5.4 Working with a supersingular surface in characteristic 2

In principle, there should be no reason why the elliptic fibration that we defined in section 5.1 could not be properly defined over a field k with characteristic 2. However, when we apply the algorithm to obtain the Weierstrass form of the fibration, we observe that it takes the form

$$Y^2 = X^3 + (t^6 + t^4 + t^2)X + (t^7 + t^5).$$

Over $k(t)$, it is easy to check that this curve has no singular points, as $t^7 + t^5$ is never a square in $k(t)$. However, when k is a perfect field in characteristic 2, every element is a square and we thus deduce that all fibres of our surface are irreducible singular rational curves. This phenomenon can only happen in characteristics 2 or 3 and, instead of referring to it as an elliptic fibration, is called a **quasi-elliptic fibration**.

Following the algorithm described in Ito's work [Ito941], which is an adaptation of Tate's algorithm for quasi-elliptic fibrations in characteristic 2, we can find the discriminant of $\mathcal{E}(k(t))$, which turns out to be

$$\Delta = t^8(t+1)^4,$$

and the reduced forms at the places 0, 1 and ∞ of this quasi-elliptic fibration, which are

$$\begin{aligned} Y^2 &= X^3 + (t^6 + t^2)X + t^7 + t^5, \\ Y^2 &= X^3 + (t+1)^6 X + (t+1)^7 + (t+1)^3, \\ Y^2 &= X^3 + (t^{-6} + t^{-2})X + t^{-7} + t^{-5}. \end{aligned}$$

From these normal forms we can deduce that in characteristic 2, $\mathcal{E}(k(t))$ still has two singular fibres of type I_4^* in $t = 0$ and $t = \infty$, whereas the two I_2 singular fibres that we got in $t = \pm 1$ in the other characteristics get merged in characteristic 2, where there is a I_0^* fibre at $t = 1$.

6 Further work

Here are some possible topics that could be interesting to do research on next, as a continuation of the theory that has been presented in this project.

6.1 Study of K3s in different ambient space

Earlier in the year, I found these examples of K3 surfaces Y, with

$$Y : \begin{pmatrix} -1 \\ 3 \end{pmatrix} \subset \left(\begin{array}{cc|ccc} 1 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right)$$

that can be described by an equation of the form $f = 0$ where f being the sum of 4 monomials, present a singular fibre of the type E_K with $6 \leq K \leq 8$ in the point $[u : v \ , \ x : y : z] = [0 : 1 \ , \ 0 : 0 : 1]$ and have fibre rank at least 12.

Singular fibres					
$[0:1\ ,\ 0:0:1]$	$[1:0\ ,\ 0:0:1]$	$[1:0\ ,\ 0:1:0]$	$[1:0\ ,\ 1:0:0]$	Fibre rank	f
E_6		A_5	A_1	12	$u^3v^2z^3 + u^2xyz + v^3y^2z + x^2y$
E_6	A_3		A_3	12	$u^3v^2z^3 + u^3y^2z + vx^2z + vxy^2$
E_6		A_6		12	$u^4vz^3 + ux^2z + vx^2z + vxy^2$
E_6	D_5	A_1		12	$u^3v^2z^3 + u^2vy^2z + v^3y^2z + x^2y$
E_6	D_5		A_1	12	$u^3v^2z^3 + uxy^2 + vx^2z + vxy^2$
E_7	A_4		A_1	12	$u^3vyz^2 + uxy^2 + v^2y^3 + vx^2z$
E_6	A_1	A_4	A_2	13	$u^3v^2z^3 + u^2xyz + vx^2z + vxy^2$
E_6	A_6		A_1	13	$u^2vxz^2 + uxy^2 + v^2y^3 + vx^2z$
E_6		A_2	A_5	13	$u^5z^3 + u^3v^2z^3 + vx^2z + vxy^2$
E_6	D_7			13	$u^3v^2z^3 + uvy^3 + v^3y^2z + x^2y$
E_6		E_7		13	$u^5z^3 + u^3v^2z^3 + v^3y^2z + x^2y$
E_7	D_5		A_1	13	$u^3v^2z^3 + uvy^3 + uxy^2 + vx^2z$
E_6	E_6		A_2	14	$u^3v^2z^3 + u^2y^3 + vx^2z + vxy^2$
E_7	D_7			14	$u^3v^2z^3 + uvy^3 + vx^2z + x^2y$
E_6		A_6	A_3	15	$u^3v^2z^3 + u^3xz^2 + vx^2z + vxy^2$
E_6	A_3	D_6		15	$u^3v^2z^3 + u^3vyz^2 + v^3y^2z + x^2y$
E_8		A_2	A_5	15	$u^5z^3 + v^2y^3 + vx^2z + vxy^2$

By analysing possible rational maps from

$$\left(\begin{array}{cc|ccc} 1 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right) \longrightarrow \mathbb{P}(3,1,1,1),$$

I have been able to construct birational maps from some of these K3 surfaces into varieties that are defined as double covers of sextics. For instance,

$$u^3v^2z^3 + u^3xz^2 + vx^2z + vxy^2 = 0 \longrightarrow s^2 + x^3y^2z + x^2yz^3 + xy^3z^2 = 0$$
$$[u : v \ , \ x, y, z] \longmapsto [s \ : \ x : y : z] = [u^2vxyz : x : uvz : u^2z]$$

$$u^3v^2z^3 + u^3vyz^2 + v^3y^2z + x^2y = 0 \longrightarrow s^2 + xy^4z + x^3yz^2 + x^2yz^3 = 0$$
$$[u : v \ , \ x, y, z] \longmapsto [s \ : \ x : y : z] = [uvxy^2 : uy : vy : uvz]$$

$$u^5z^3 + v^2y^3 + vx^2z + vxy^2 = 0 \longrightarrow s^2 + sz^3 + x^2y^3z + x^3y^2z = 0$$
$$[u : v \ , \ x, y, z] \longmapsto [s \ : \ x : y : z] = [uvx^2z : x : vy : u^2z]$$

It would be interesting to study these K3 surfaces defined as double covers of sextics, as there is an easy way of constructing divisors as the pullback of plane lines that are tritangent to the branch locus [FvS189].

Also, moving towards more intricate varieties, it would be nice to study 3-folds endowed with elliptic fibrations which could be constructed from the examples of K3 surfaces that we have found.

6.2 Analysis of other aspects of our supersingular surface X

In this project we have only studied one elliptic fibration of X. However, the lines that we found on section 5.1 to study the Picard rank, can allow us to construct other elliptic fibrations, that will likely present different trivial lattices, Mordell-Weil lattices and torsion. Studying those could be interesting as we might find fibrations in which the Mordell-Weil group has a high rank, and thus, by specialising in certain fibres, we could build explicit examples of elliptic curves with high ranks, as in [SS19, Chapter 13].

A similar problem but with a different flavour would be counting all possible types of elliptic fibrations on X. This goal can be achieved by studying which possible lattices can arise as sublattices of $\mathrm{NS}(X)$, as shown in the work of Bertin et al. [BGH$^+$15].

Another alternative would to study specific structures that are associate to X for its condition of supersingular surface. An example of this would be analysing its **Shioda-Inose structure**, that is, how to construct a rational map of degree two to some Kummer surface in a way that the **transcendental** lattice $T(X)$ is preserved, which is the image of $\mathrm{NS}(X)^\perp$ inside of $H^2(X, \mathbb{Z})$ [SS097, Section 13.4].

Finally, it is important to mention that every singular K3 surface X over $\mathbb{Q}$ is modular: the Galois representation on $T(X)$ is associated to a Hecke eigenform of weight 3. Not only that, but most of the known Hecke eigenforms of weight 3 with eigenvalues $a_p \in \mathbb{Z}$ are associated to a singular K3 surface over $\mathbb{Q}$ [SS097, Section 13.13], so it could be very formative to study this correspondence in an explicit example.

7 Magma code

7.1 Geometric analysis of the surface X

```
k := Rationals();
P3<x,y,z,w> := ProjectiveSpace(k,3);
X := Surface(P3,x^3*y + x*y^2*w + z*y*w^2 + z^3*w);
X;
GeometricGenus(X);
ArithmeticGenus(X);
Irregularity(X);
[ChernNumber(X,i) : i in [1,2]];
for i in [0..2], j in [0..2] do
printf "%o,%o : %o ",i,j,HodgeNumber(X,i,j);
end for;
IsNormal(X);
HasOnlySimpleSingularities(X : ReturnList := true);
KodairaEnriquesType(X);
```

7.2 Construction of an elliptic fibration

```
Q<t> := FunctionField(Rationals());
P<x,y,w> := ProjectiveSpace(Q,2);
C := Curve(P, x^2*y+y^2*w+t*y*w^2+t^3*x^2*w);
pt := C![1,0,0];
E := MinimalDegreeModel(EllipticCurve(C,pt));
TorsionSubgroup(E);
LocalInformation(E);
Discriminant(E);
Conductor(E);
jInvariant(E);

K<t> := FunctionField(GF(3));
P<x,y,w> := ProjectiveSpace(K,2);
C := Curve(P, x^2*y+y^2*w+t*y*w^2+t^3*x^2*w);
pt := C![1,0,0];
E := MinimalDegreeModel(EllipticCurve(C,pt));
E;
TorsionSubgroup(E);
LocalInformation(E);
Discriminant(E);
Conductor(E);
jInvariant(E);
```

7.3 Intersection matrices

```
k := Rationals();
P3<x,y,z,w> := ProjectiveSpace(k,3);
X := Surface(P3,x^3*y+x*y^2*w+z*y*w^2+z^3*w);
R := CoordinateRing(P3);
L0 := Divisor(X,ideal<R|x,z>);
H := Divisor(X,ideal<R|x>);
D0 := H - L0;
L1 := Divisor(X,ideal<R|y,w>);
L2 := Divisor(X,ideal<R|x-z,y+w>);
L3 := Divisor(X,ideal<R|x+z,y-w>);
L4 := Divisor(X,ideal<R|y,z>);
L5 := Divisor(X,ideal<R|x,w>);
D6 := Divisor(X,ideal<R|x-z,x^2+y*w>);
D7 := Divisor(X,ideal<R|y+w,z^2+x*z+x^2-w^2>);
D8 := Divisor(X,ideal<R|x+z,x^2+w*y>);
D9 := Divisor(X,ideal<R|y-w,x^2+z^2-x*z+w^2>);
D10 := Divisor(X,ideal<R| x*y-w*z,x^2+2*y*w+z^2>);

dlist := [*D0, L1, L2, L3, L4, L5, D6, D7, D8, D9, D10*];
dlist;
mat := Matrix(Rationals(), #dlist, #dlist, [<i, j,
    IntersectionNumber(dlist[i],dlist[j])>:i,j in [1..#dlist]]);
mat;
Rank(mat);
shlist := [*D0, L1, L2, L3, L4, L5, D6, D7, D10*];
Rank(Matrix(Rationals(), #shlist, #shlist, [<i, j,
    IntersectionNumber(shlist[i],shlist[j])>:i,j in [1..#shlist]]));

nlist := [*L1, L2, L3, D6*];
nmat := Matrix(Rationals(), #nlist, #nlist, [<i, j,
    IntersectionNumber(nlist[i],nlist[j])>:i,j in [1..#nlist]]);
nmat;
Rank(nmat);
sing,typ := ResolveSingularSurface(X);
sing;
s1:=ChangeRing(IntersectionMatrix(sing[1]),Rationals());
s2:=ChangeRing(IntersectionMatrix(sing[2]),Rationals());

lat := DirectSum(DirectSum(nmat,s1),s2);
lat;
Rank(lat);
Determinant(lat);
```

7.4 Computation of an upper bound of the Picard rank

```
p := 2;
N := 10;
sign := 1;
L1 := [];
L2 := [];
for n in [1 .. N] do
   P<x,y,z,w> := ProjectiveSpace(GF(p^n),3);
   X := Surface(P, y*x^3 + x*y^2*w + z*w^2*y + w*z^3);
   pts := #[pt: pt in Points(X)];
   Append(~L1, pts);
   Append(~L2, pts-p^(2*n)-1);
end for;
L1;
L2;
L3 := [-L2[1]];
for k in [2 .. N] do
   count := 0;
   for i in [1 .. k-1] do
      count := count+ L3[i]*L2[k-i];
   end for;
   Append(~L3,(L2[k]+count)/(-k));
end for;
for j in [1 .. N] do
   Append(~L3, sign*L3[N+1-j]*p^(2*j));
end for;
Append(~L3, sign*p^(2*N+2));
L3;
if sign eq -1 then
   Insert(~L3, N+1, 0);
else
   count := p^(2*N+2);
   for i in [1 .. N] do
      count := count + L3[i]*p^(2*N+2-i);
   end for;
for i in [1 .. N+1] do
   count := count + L3[N+i]*p^(N+1-i);
end for;
Insert(~L3, N+1, count);
end if;
L3;
P<t> := PolynomialRing(Rationals());
f := t^22+ Polynomial(Reverse(L3));
g := p^(-22)*Evaluate(f, p*t);
Factorisation(g);
```

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